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The Application of Lagrange and Pidduck-Kent Gradient Models to Guns Using Low Molecular Weight Gases

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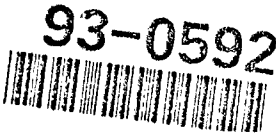

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13. ABSTRACT *(Maximum 200 words)*

Increasing interest in hypervelocity and the potential offered by electrothermal gun systems for the practical application of novel propellants have led to a search for energetic materials which, upon reaction, generate gaseous products with low average molecular weight. It has been suggested that, as a result of using low molecular weight gases, traditional interior ballistic gradient models are not applicable in simulations of the interior ballistic process.

The assumptions underlying the classical interior ballistic gradient models are reviewed and their ranges of applicability are discussed. Comparisons of the results of lumped parameter interior ballistic simulations using the Lagrange and Pidduck-Kent pressure gradient models and a one-dimensional hydrodynamic model, over a range of charge to mass ratios, are presented. It is shown that the traditional gradient models are applicable in the simulation of a gun using a low molecular weight gas, and are thus applicable in simulations of the electrothermal gun, within the normal limits of validity of the gradient models.

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1. INTRODUCTION

Gradient models are analytic representations of the spatial distributions of density, pressure, temperature, and velocity of expanding propellant gases between the gun breech and the projectile base. In a thermodynamic interior ballistic model, only mass and energy conservation are explicitly required, and, therefore, a gradient model must be provided to account for the spatial variation in gas momentum. Closure is achieved by including a constitutive law for the gas and requiring that, at any point in time during the interior ballistic process, an integral average of the local thermodynamic properties of the gas equal the corresponding average gas properties derived from the constitutive law and energy conservation for the system as a whole.

In general, use of a gradient model results in fixed gradients of system properties throughout the interior ballistic process and, thus, a fixed ratio of projectile base pressure to breech pressure (Corner 1950; Vinti and Kravitz 1949). Exceptions arise if boundary conditions and ratio of propellant charge mass to projectile mass (i.e., charge to mass ratio [C/M]), vary during the interior ballistic process, as in the Modified Lagrange gradient model of Morrison and Coffee (1990), or if multiphase effects are included, as in the RGA gradient model of Robbins, Anderson, and Gough (1990). The fixed (or constant) gradient character of basic gradient models is a result of the exclusion of wave phenomena, such that changes in the physical state of the system are instantaneously communicated throughout the gas in accordance with the distribution functions of gas properties for a particular gradient model. Thus, gradient models entail an implicit assumption of infinite gas sound speed.

The development of gradient models has been closely tied to the ballistics of conventional guns, and such models do not appear to have been widely applied to novel propulsion concepts such as the light gas gun (Seigel 1979). However, in the derivation of traditional gradient models, i.e., the Lagrange (Corner 1950) and the Pidduck-Kent (Vinti and Kravitz 1949) models, there are no assumptions which limit their application to conventional guns or to gas properties characteristic of conventional gun propellants. Therefore, traditional gradient models should be applicable in thermodynamic interior ballistic models of gas dynamic propulsion processes irrespective of molecular weight or sound speed of the driving gas, subject to the usual consideration of expansion ratio and, in the case of the Lagrange

gradient, C/M (C[illegible] 1950; Vinti and Kravitz 1949). Indeed, the accuracy of a gradient model for a give[illegible] configuration should improve with increasing sound speed since wave transit time in the[illegible]olumn is reduced. As a result, the relaxation of wave processes which dominate early ex[illegible]n and the subsequent establishment of a nearly constant base-to-breech pressure ra[illegible]ld occur more quickly, thus, at a lower expansion ratio for a low molecular weight ga[illegible]sing the range of validity and the accuracy of the gradient model. Thus, the pressure [illegible]pproximation and the accuracy of ballistic solutions obtained using traditional grad[illegible]els should actually improve as molecular weight is decreased and sound speed inc[illegible]es.

It has been suggest[illegible]that the electrothermal gun concept may offer a practical means to employ low molecular wei[illegible]t gases (Juhasz et al. 1988). In the case of an inert working fluid, it has been experimentally [illegible]monstrated that the rate of conversion of liquid to gas, as well as the pressure, density, and [illegible]perature of the gas, are dependent on the temporal and spatial distributions of the plasma s[illegible]e. While experimental data does suggest some success in achieving ballistic control and r[illegible]eatability with inert working fluids, the data does not constitute a clear demonstratio[illegible] ballistic control. In the case of energetic working fluids, or propellants, the evidence of a c[illegible] correspondence between the rate of conversion of propellant to gas and the charac[illegible]stics of the plasma source is less conclusive, particularly for cases in which the ratio of propellant chemical energy to plasma energy is greater than about 2 to 4. Although experimental data is somewhat limited, it would appear that reasonable ballistic repeatability may be achievable for ratios of propellant chemical energy to plasma energy at least on the order of one.

If, however, it is assumed that for some combinations of plasma source and working fluid (or propellant), gas generation can be precisely controlled by the plasma source such that variations in the rate of plasma deposition will result in corresponding variations in chemical energy release, then repeatable control of the ballistic process may also be assumed. If it is further assumed that such permissible combinations of plasma source and working fluid are possible for a sufficiently wide range of working fluid characteristics, it may then be possible to identify unconventional propellants, which upon reaction generate low (average) molecular weight gaseous products. In comparison with the combustion products of conventional solid propellants, a low molecular weight gas contains a greater number of molecules per unit

mass, the ratio of the number of moles per unit mass being inversely proportional to the ratio of molecular weights of the two gases. As a result, for a given gas temperature, the low molecular weight gas will have higher specific internal energy and impetus, as well as higher speed of sound, such that a lower mass of propelling gas is required to achieve a given initial gas internal energy. Therefore, for fixed initial gas internal energy, the kinetic energy of a low molecular weight propelling gas, at any given projectile velocity, represents a smaller fraction of the total chemical energy of the system, resulting in a reduced projectile base to breech pressure ratio (after relaxation of wave processes) and increased thermodynamic efficiency of the expansion process (i.e., higher ballistic efficiency than is achievable with conventional propellants).

In recent discussions of the electrothermal-chemical gun concept, it has been suggested that the reduced pressure gradient in a gun using a low molecular weight gas is a result of the higher sound speed of the gas, which permits more rapid transit of pressure waves between the breech and the projectile base. Furthermore, it has been suggested that traditional gradient models, which exclude wave phenomena, should not, therefore, be used in interior ballistic simulations of the electrothermal gun. As noted earlier, the derivations of such gradient models do not introduce limitations on the properties of propelling gases. Thus, it would appear that their application should not be limited to the conventional gun case.

The objective of this investigation is to computationally explore the range of applicability of the standard gradient models, Lagrange and Pidduck-Kent, in the simulation of interior ballistic processes involving gases of varying molecular weight. To simplify the analysis, the Lagrange ballistic problem is used as the framework for simulations. The lumped parameter IBHVG2 computer model (Anderson and Fickie 1987) was used to perform interior ballistic simulations with the Lagrange and Pidduck-Kent gradient models. The one-dimensional XNOVAKTC computer model (Gough 1990) was used to provide wave dynamic solutions as a baseline for comparisons. The results indicate that traditional gradient models are indeed applicable in interior ballistic simulations of gun concepts utilizing low molecular weight gases.

2. LAGRANGE BALLISTIC PROBLEM

The Lagrange problem is an idealization of the interior ballistic process in a gun. It is assumed that the gun is a right circular cylinder (i.e., chamber and bore diameter are equal)

closed at one end by the breech. It is also assumed that prior to the start of projectile motion, the propellant is instantaneously burned, such that the chamber is initially filled with a gas of uniform pressure, density, and temperature. The gas is assumed to be inviscid and heat loss to the walls is neglected.

Corner (1950) has provided a historical summary of efforts to solve the Lagrange problem, including the development of the Lagrange and Pidduck-Kent gradient models. These models are addressed in greater detail in subsequent sections and in Appendices A and B. The first wave dynamic solution of the Lagrange problem was presented by Love and Pidduck (1922). They analyzed the Lagrange problem using the method of characteristics, tracing the rarefaction wave generated by acceleration of the projectile, as it moves through the gas column between the projectile base and the breech. More recently, the Lagrange problem and the solution of Love and Pidduck (1922) have been used as a test case for multidimensional interior ballistic models (Schmitt and Mann 1981; Edelman, private communication).

The pressure distributions behind the projectile at various times during the ballistic process, as obtained by Love and Pidduck (1922), are shown in Figure 1. The initial chamber pressure is 6,333.3 kg/cm^2 or 621.09 MPa. As the projectile begins to accelerate, a rarefaction wave is generated at the projectile base and begins to move toward the breech. The pressure gradient between the projectile base and the wave front prior to the first wave reflection from the breech, and, thus, the gas kinetic energy distribution, are characteristic of a simple wave. The wave reflects from the breech as a rarefaction, producing a nearly flat pressure distribution and a base-to-breech pressure ratio near 1.0 when the wave reaches the projectile base. This reduced pressure gradient results in reduced gas acceleration and, thus, a reduced rate of increase of gas kinetic energy. Upon reflection of the rarefaction wave from the projectile base, the process repeats; however, the pressure gradient is substantially reduced. By the second reflection from the projectile base, the wave disturbance has relaxed significantly, and, as can be seen in Figure 2, the ratio of base-to-breech pressure is approaching a constant value of approximately 0.9.

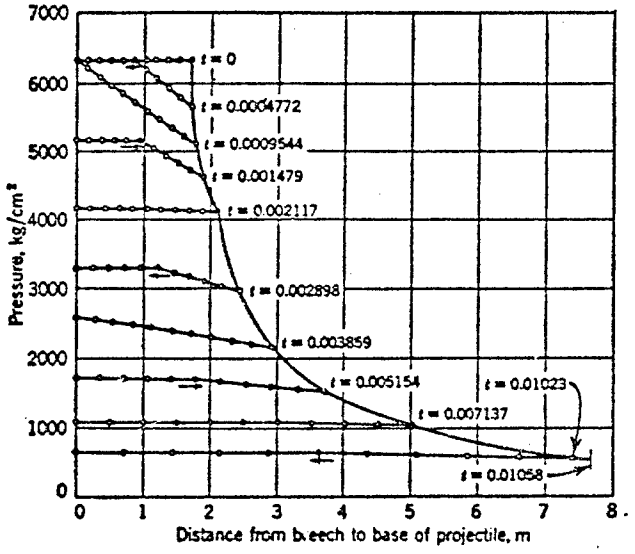

Figure 1. Pressure Distribution Behind the Projectile at Various Stages of Its Motion, in the Example Studied by Love and Pidduck (1922).

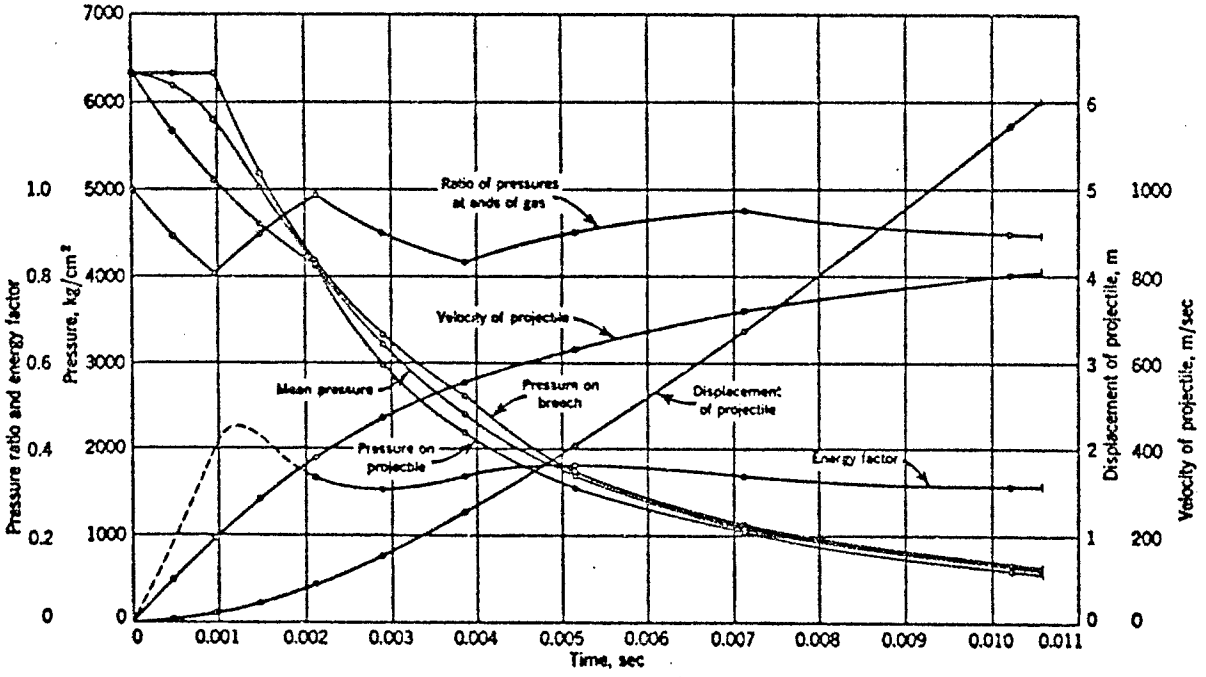

Figure 2. Time Variation of Results, in the Example Studied by Love and Pidduck (1922).

3. LAGRANGE GRADIENT MODEL

The Lagrange gradient model is the instantaneous solution of the continuity and momentum equations for a gaseous propelling charge in the Lagrange gun, with the assumption that the gas density is uniform between the breech and the projectile base (see Appendix A). This leads to a gas velocity distribution which varies linearly with position along the tube, from zero at the breech to the projectile velocity at the base of the projectile. The resulting pressure distribution is a quadratic function of distance from the breech. As noted by Corner (1950), the term "Lagrange approximation" is applied to both the assumption of zero gas density gradient and the assumption of a linear gas velocity distribution. It can be shown that a zero gas density gradient implies a linear gas velocity distribution, but the reverse is not necessarily true. While the Lagrange model is an approximate solution to the Lagrange ballistic problem, it must be noted that this model has been routinely applied in the simulation of "real" guns in which the propellant combusts at a finite rate during the ballistic process. In this case, it is assumed that the unburnt propellant is uniformly distributed throughout the gas, moves at the local gas velocity, and burns at a rate dependent on the space mean pressure.

The Lagrange gradient model is generally applicable in cases for which the charge-to-mass ratio is "small." However, reasonable results are obtained for C/M in excess of 2.0. Using the velocity and pressure distributions of the Lagrange model, relationships between breech, space mean, and base pressures, as well as gas and projectile kinetic energies, are obtained. These are

$$P_{BREECH} = P_{BASE}\left(1 + \frac{\varepsilon}{2}\right) \tag{1}$$

$$\bar{P} = P_{BASE}\left(1 + \frac{\varepsilon}{3}\right) \tag{2}$$

$$KE_{GAS} = KE_{PROJ}\left(\frac{\varepsilon}{3}\right) \tag{3}$$

where

$$\varepsilon = \frac{C}{M} \tag{4}$$

is the ratio of propellant mass to projectile mass and $\bar{P}$ is the space mean pressure of the gas. The gradients of pressure and velocity in the Lagrange model are not dependent on specific gas properties, other than C/M, and the numerical factors which arise in the model (1/2 and 1/3) are constants independent of system parameters.

4. PIDDUCK-KENT GRADIENT MODEL

The Pidduck-Kent model is a special solution of the continuity and momentum equations for the gaseous propelling charge in the Lagrange gun. This special solution was first published by Love and Pidduck (1922). A more detailed presentation was latter published by Kent (1936) and, subsequently, an extended treatment was published by Vinti and Kravitz (1949). The Pidduck-Kent model addresses the nonuniformity of gas density which is neglected in the Lagrange model and which becomes increasingly important with increasing C/M. This is accomplished by assuming that each element of the gas evolves adiabatically from its initial state with all gas elements following the same adiabatic. In the resulting solution, it is found that, in its initial state, the gas density, pressure, and temperature are decreasing functions of distance along the tube from the breech to the projectile base. A detailed derivation of the Pidduck-Kent solution is presented in Appendix B. Pressure and energy relationships analogous to Equations 1–3 are obtained in the Pidduck-Kent gradient model. These are

$$P_{BREECH} = P_{BASE}(1 - a_0)^{-(n+1)} \tag{5}$$

$$\bar{P} = P_{BASE}\left(1 + \frac{\varepsilon}{\delta}\right) \tag{6}$$

$$KE_{GAS} = KE_{PROJ}\left(\frac{\varepsilon}{\delta}\right) \tag{7}$$

where n is the polytropic index,

$$\frac{1}{(\gamma - 1)},$$

and a_0 and δ are parameters arising in the Pidduck-Kent solution which are related by

$$\frac{1}{\delta} = \frac{1}{2n+3}\left[\frac{1}{a_0} - \frac{2(n+1)}{\varepsilon}\right]. \tag{8}$$

In the limit as ε (i.e., C/M) approaches zero, δ approaches 3, and a_0 approaches

$$\frac{\varepsilon}{[2(n+1)]}$$

(see Appendix B) such that Equations 5–7 reduce to the Lagrange model, Equations 1–3.

In contrast with the Lagrange model, the parameters of the Pidduck-Kent solution, a_0 and δ, are functions of C/M and the ratio of specific heats. Similarly, the distributions of thermodynamic quantities are dependent on both C/M and the ratio of specific heats. The calculation of a_0 and δ for a specific ballistic system has been simplified by the development of a least squares fit of computed values of δ using a polynomial function in ε and γ (Grollman and Baer 1970). (The least squares fit of δ was completed for $1.2 \leq \gamma \leq 1.3$, however, values of δ for $\gamma = 1.4$ are within 1–2% of values obtained using the tables of Vinti and Kravitz [1949], see Appendix D). Equation 8 is then used to calculate a_0.

It has been generally supposed, though never proven, that the Pidduck-Kent solution of the Lagrange problem approaches the wave dynamic solution for sufficiently large projectile travel. Additional discussion of this issue can be found in Appendices B and E.

5. PROBLEM DESCRIPTION

The Lagrange ballistic problem was chosen as the framework for evaluation of the range of applicability of the Lagrange and Pidduck-Kent gradient models. This physically simple configuration was chosen to facilitate direct comparison of lumped parameter solutions based on these gradient models with one-dimensional wave dynamic solutions. Not only is the Lagrange problem the framework for the development of these gradient models, but, by

assuming a preburned propellant, differences in lumped parameter and wave dynamic solutions arising from physical models of component processes (i.e., ignition, combustion, drag, etc.) are avoided.

In order to investigate the effects of gas molecular weight, three propelling gas compositions were selected. These are hydrogen gas, which has been heated by an external source; the combustion products of a mixture of 100% hydrogen peroxide and octane with an oxidizer-to-fuel ratio of approximately 10 to 1; and the combustion products of JA2 solid propellant. The thermochemical properties of these gases, calculated using the Blake code (Freedman 1982) are given in Table 1. The initial gas pressure was chosen to be 500 MPa in all cases. The initial temperature of the hydrogen gas was chosen to be the flame temperature of JA2, while the flame temperature of the octane-hydrogen peroxide mixture is somewhat lower.

The constant bore diameter of the Lagrange gun was chosen to be 120 mm. The projectile mass is 9.0 kg. Calculations were performed for the C/M of 0.5, 1.0, 3.0, and 8.0 with the gas properties presented in Table 1. This leads to different initial chamber volumes in each case as indicated in Table 2. Shot start pressure, bore resistance, and gas pressure ahead of the projectile were assumed to be zero and heat transfer was neglected in all calculations.

The XNOVAKTC code (Gough 1990) was used for one-dimensional simulations of the Lagrange gun, while the IBHVG2 code (Anderson and Fickie 1987) was used in thermodynamic simulations with both the Lagrange and Pidduck-Kent (Grollman and Baer 1970) gradient model options of this latter code. It is noted that the analytic solution to the Lagrange problem, as presented in Appendices A (Lagrange Gradient) and B (Pidduck-Kent Gradient), could have been used in place of IBHVG2. However, this latter computer model has been extensively tested and is readily accessible.

Table 1. Propelling Gas Properties

	Hydrogen	Octane/ Peroxide	JA2
Initial Pressure (MPa)	500	500	500
Initial Temperature (K)	3,410	3,017	3,410
Initial Density (g/cm^3)	3.1204×10^{-2}	2.4892×10^{-1}	3.0465×10^{-1}
Ratio of Specific Heats	1.4	1.2348	1.225
Covolume (cm^3/g)	3.695	1.044	0.996
Molecular Weight (g/mole)	2.0	16.873	24.8
Impetus (J/g)	14,175.4	1,486.7	1,140
Specific Energy (J/g)	35,438.5	6,331.8	5,066.7
Gas Sound Speed (m/s)	5,036	1,831	1,699
Ideal Gas Sound Speed (m/s)	4,454	1,355	1,183

Table 2. Chamber Volumes

C/M	Mass of Gas (kg)	Chamber Volume (liters)		
		Hydrogen	Octane/ Peroxide	JA2
0.5	4.5	144.2	18.08	14.74
1.0	9.0	288.4	36.16	29.48
3.0	27.0	865.2	108.48	88.44
8.0	72.0	2,307.2	289.28	235.84

6. RESULTS AND DISCUSSION

The results of the interior ballistic simulations are presented as graphs of dimensionless projectile velocity ($\overline{u_p}$) vs. dimensionless projection travel ($\overline{x_p}$, see Equations E-21 and E-22) for C/M of 0.5, 1.0, 3.0, and 8.0.* The trajectories for the hydrogen propelling gas are presented in Figures 3 and 4. At a C/M of 0.5, the Lagrange and Pidduck-Kent trajectories coincide with the trajectory obtained from the one-dimensional simulation. At C/M of 1.0, the one-dimensional and Pidduck-Kent trajectories are nearly indistinguishable, however, the Lagrange trajectory has begun to diverge. At C/M of 3.0, the Lagrange trajectory has clearly diverged from the one-dimensional trajectory by about 5% for dimensionless projectile travels greater than about 15, which corresponds to an expansion ratio,

$$1 + \left(\frac{x_p}{x_0}\right),$$

of 7.19 (see Table 3). From Equation 3, we see that, for C/M = 3.0, the Lagrange model predicts that the gas and projectile kinetic energies are equal. In contrast, the Pidduck-Kent model (Equation 7) predicts that the gas kinetic energy is about 85% of the projectile kinetic energy (see Table 4). This means that the Lagrange model over-predicts the gas kinetic energy for large C/M, resulting in an under-prediction of projectile velocity.

The Pidduck-Kent trajectory initially diverges from the one-dimensional trajectory for dimensionless projectile travels between 1 and 10 (C/M = 3.0); however, the two trajectories then converge for $\overline{x}_p > 20$. This behavior is related to the wave dynamics in the gas during the early portion of the ballistic process (i.e., low expansion ratios). The energy partition for the one-dimensional solution, for C/M = 3.0, is presented in Figure 5 as a function of dimensionless projectile travel. During the early travel, the wave dynamics of the system are reflected in the gas kinetic energy. However, the key feature of the energy partition is the relationship between gas and projectile kinetic energies. Up to a dimensionless travel of

*The introduction of dimensionless variables facilitates the comparison of results for propelling gases of substantially different sound speeds and molecular weights. The specific choice of dimensionless variables is suggested by Siegel (1979) based on a characteristic solution of the preburnt propellant ideal gas gun.

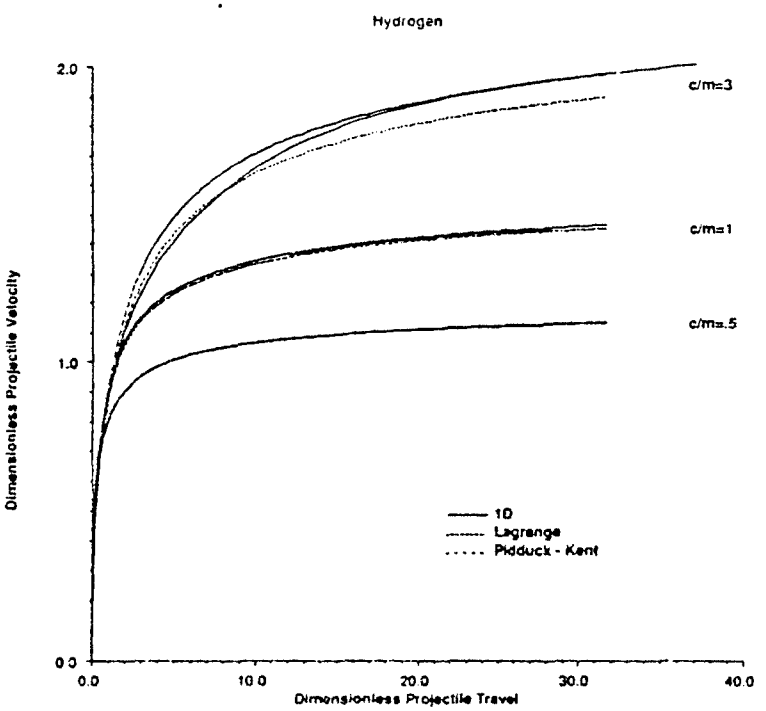

Figure 3. Projectile Trajectories for Hydrogen Gas; C/M = 0.5, 1.0, and 3.0.

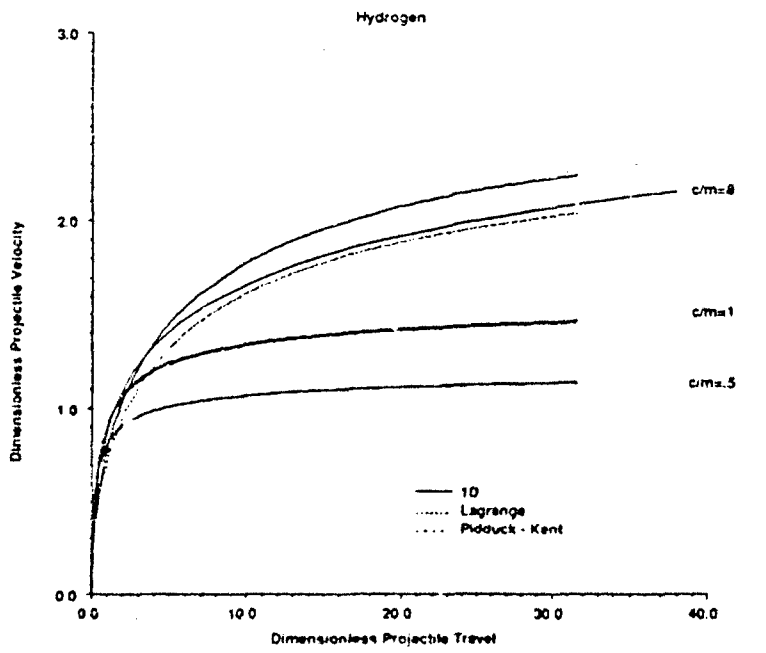

Figure 4. Projectile Trajectories for Hydrogen Gas; C/M = 0.5, 1.0, and 8.0.

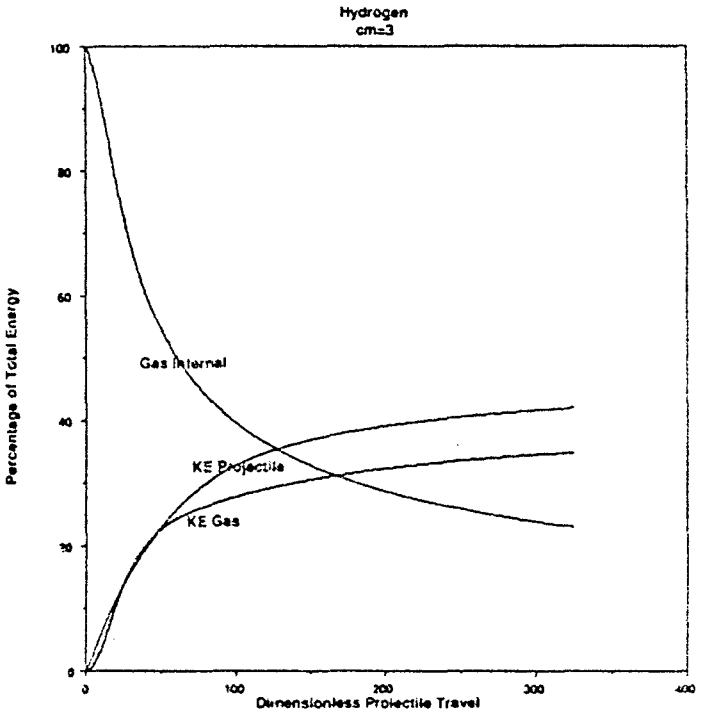

Figure 5. Energy Partition for Hydrogen Gas; C/M = 3.0.

Table 3. Relationship Between Expansion Ratio and Dimensionless Projectile Travel for Hydrogen

$\bar{x}_p$	Expansion Ratio			
	C/M = 0.5	C/M = 1.0	C/M = 3.0	C/M = 8.0
5	13.39	7.19	3.06	1.77
10	25.77	13.39	5.13	2.55
15	38.16	19.58	7.19	3.32
20	50.54	25.77	9.26	4.10
25	62.93	31.96	11.32	4.87
30	75.32	38.16	13.39	5.64
40	100.09	50.54	17.51	7.19
50	124.86	62.93	21.64	8.74
100	248.72	124.86	42.49	16.48

Table 4. Pidduck-Kent and Lagrange Parameters; $\gamma = 1.4$

			KE_{GAS}/KE_{PROJ}			P_{BREECH}/P_{BASE}		
C/M	δ	a_0	C/δM	C/3M	1 – D	$(1 - a_0)^{-(n+1)}$	1 + c/2M	1 – D
0.5	3.1272	0.0604	0.1599	0.1667	0.1591	1.2436	1.25	1.2543
1.0	3.2315	0.1055	0.3095	0.3333	0.3137	1.4775	1.50	1.4848
3.0	3.5288	0.2174	0.8502	1.0000	0.8281	2.3581	2.50	2.3412
8.0	3.9482	0.3447	2.0262	2.6667	2.0705	4.3895	5.00	4.3156

Note: Pidduck-Kent parameters, δ and a_0, are obtained from BRLVG2 calculations based on Grollman and Baer (1970) least-squares fit.

about 10, the gas and projectile kinetic energies are approximately equal. Above 10, they diverge and approach a nearly constant ratio of gas to projectile kinetic energy of approximately 0.83, which is reflected in the projectile trajectory (see Figure 3). Similar results are observed for the C/M = 8.0 case in Figures 4 and 6. However, the gas kinetic energy is greater than the projectile kinetic energy in this case and, due to the substantially increased initial chamber length, a dimensionless travel of 30 is not quite in the asymptotic (isentropic) region of the projectile trajectory. However, from Figure 6, it can be seen that the ratio of gas to projectile kinetic energy approaches a constant value, for large dimensionless travel, which is close to the value obtained with the Pidduck-Kent model (see Table 4). We also note that, despite the early behavior of the Lagrange trajectory (i.e., the Lagrange and one-dimensional trajectories are quite close for dimensionless travels up to 30), the Lagrange trajectory diverges from the one-dimensional solution for larger expansion ratios as the Pidduck-Kent and one-dimensional solutions converge.

The results for JA2 and the octane-hydrogen peroxide mixture are presented in Figures 7–10. In general, these results are very similar to those already discussed for the hydrogen case when presented in terms of dimensionless projectile travel and velocity. All trajectories are nearly identical for C/M of 0.5 and 1.0. For C/M = 3.0, the Pidduck-Kent trajectories diverge above the one-dimensional trajectories during early travel and converge for $\bar{x}_p > 25$, while for C/M = 8.0 and $\bar{x}_p > 30$, the Pidduck-Kent solutions have not reached the asymptotic region and lie above the one-dimensional trajectories. The Lagrange trajectories

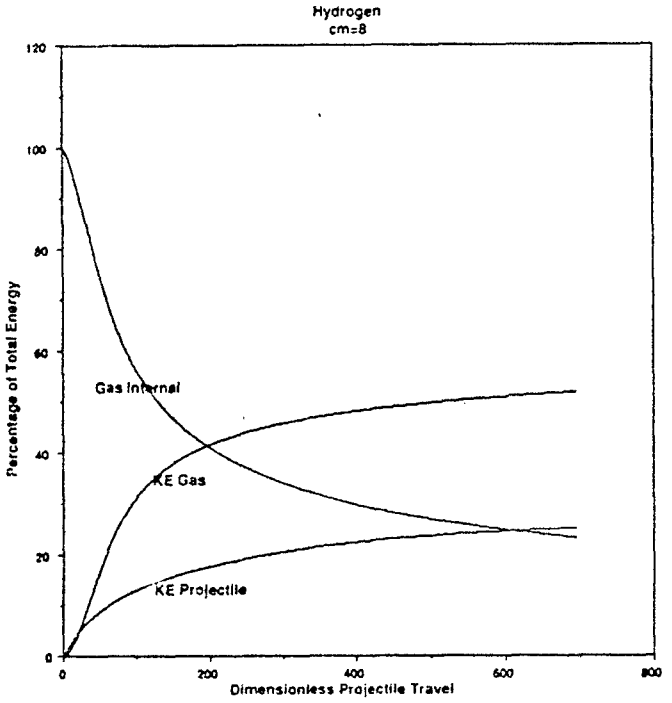

Figure 6. Energy Partition for Hydrogen Gas; C/M = 8.0.

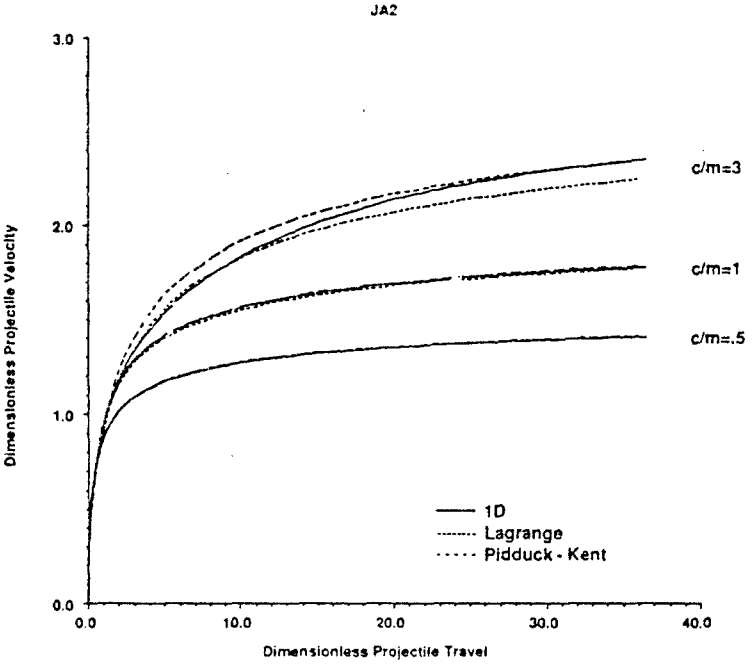

Figure 7. Projectile Trajectories for JA2 Combustion Products; C/M = 0.5, 1.0, and 3.0.

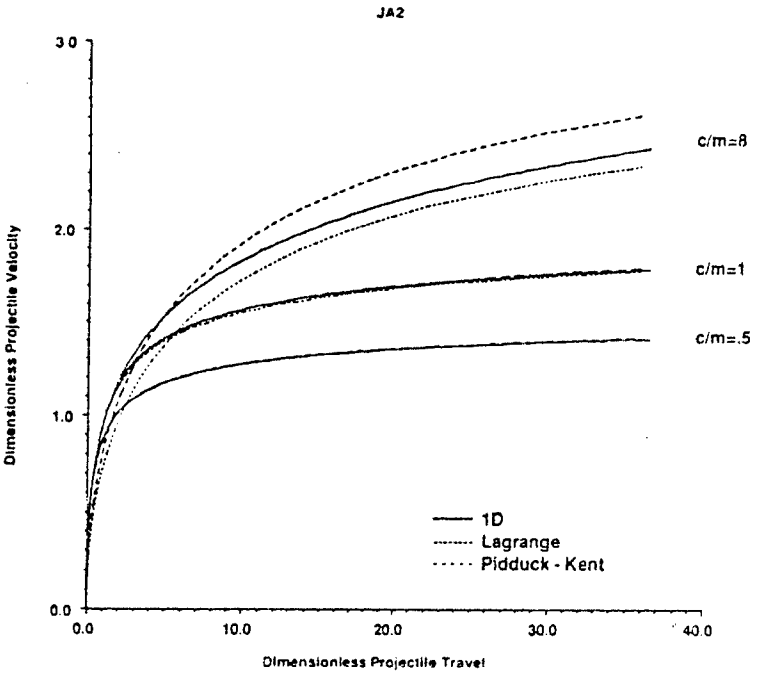

Figure 8. Projectile Trajectories for JA2 Combustion Products; C/M = 0.5, 1.0, and 8.0.

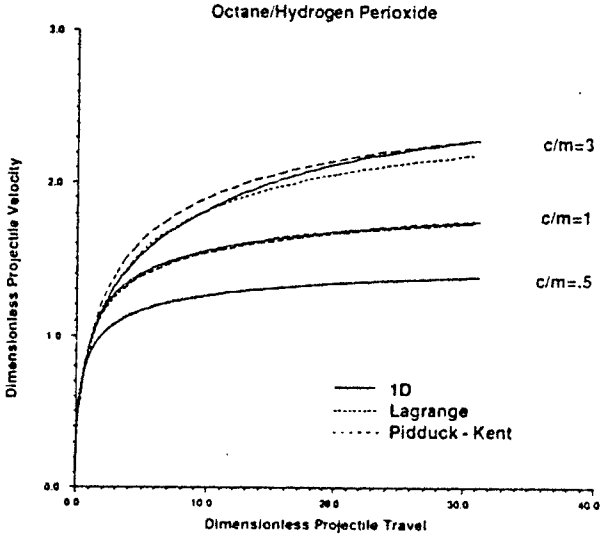

Figure 9. Projectile Trajectories for Octane/Hydrogen Peroxide Combustion Products; C/M = 0.5, 1.0, and 3.0.

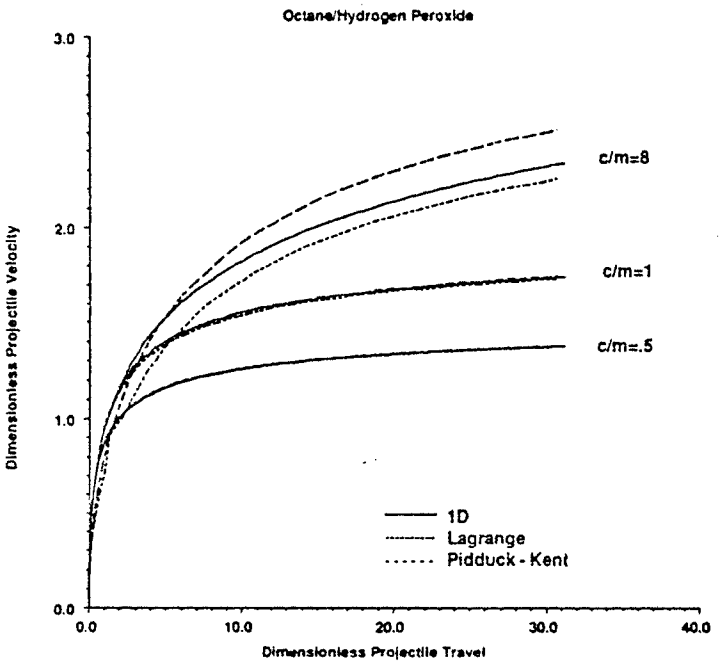

Figure 10. Projectile Trajectories for Octane/Hydrogen Peroxide Combustion Products; C/M = 0.5, 1.0, and 8.0.

for C/M of 3.0 and 8.0 lie below the one-dimensional trajectories and, in the C/M = 8.0 case, the Lagrange solutions are expected to diverge with increasing travel.

7. CONCLUSIONS

The applicability of the Lagrange and Pidduck-Kent gradient models in the simulation of the interior ballistic process for gases of various molecular weights has been explored using the Lagrange ballistic problem as the basic framework for the investigation. The results indicate that gas molecular weight and sound speed do not impact the validity of these gradient models, as was anticipated from consideration of the theoretical development of these models. All computed trajectories for C/M of 0.5 and 1.0 are nearly identical. The Lagrange model appears to provide reasonably accurate results up to C/M of about 3.0 (e.g., velocities within about 5% of the one-dimensional solution at C/M = 3.0 with increasing accuracy for lower C/M). Using the Pidduck-Kent model, the calculated projectile velocity approaches the one-dimensional solution with increasing projectile travel for C/M of 3.0 and

8.0. The results also indicate that the Pidduck-Kent and one-dimensional solutions converge for expansion ratios characteristic of real guns.

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APPENDIX A:

THE LAGRANGE GRADIENT MODEL

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The Lagrange gradient model was developed as a solution to the Lagrange ballistic problem, which is described in the text and in Appendix B. This gradient model is widely used in interior ballistic simulations of real guns, for which the original assumption of a preburned propellant gas must be relaxed. While the assumptions used in the derivation of the Lagrange gradient model are valid only for low charge-to-mass ratio (C/M), it has been found in practical applications that the model provides reasonable accuracy for C/M as high as 3.0 (e.g., velocities within about 5% of a one-dimensional solution at C/M = 3.0 with increasing accuracy for lower C/M).

A derivation of the Lagrange gradient model is presented here primarily for completeness. However, concepts used in the derivation of the Lagrange gradient are also applied in the development of the Pidduck-Kent solution in Appendix B. Therefore, inclusion of the derivation provides an easy reference for the reader. This particular derivation of the Lagrange gradient model was previously published by Morrison and Coffee (1990).

The equations of motion governing the motion of the gas in a constant area gun in region from breech to projectile base are

$$\frac{\partial \rho}{\partial t} + v\frac{\partial \rho}{\partial x} + \rho\frac{\partial v}{\partial x} = 0 , \tag{A-1}$$

$$\rho\left[\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x}\right] = -\frac{\partial P}{\partial x} , \tag{A-2}$$

with the boundary conditions

$$x_{BREECH} = 0 , \tag{A-3}$$

$$x_{BASE} = y \tag{A-4}$$

$$v_{BREECH} = 0 , \tag{A-5}$$

$$v_{BASE} = \frac{dy}{dt} = u_p \,, \tag{A-6}$$

where u_p is the velocity of the projectile.

In the development of the Lagrange pressure gradient "it is assumed all the propellant charge (C) is in gaseous form at the time considered." However, "the theory applies without alteration if it is assumed that," prior to consumption of all the propellant charge, "the unburnt charge moves with the gas, the distribution of the solid along the bore being the same as the distribution of gas (Corner 1950)." The gas is assumed to be inviscid and heat loss to the walls is neglected such that the flow is isentropic.

We now assume that the density of the gas (or gas plus unburnt charge) is uniform over the region behind the projectile, i.e.,

$$\frac{\partial \rho}{\partial x} = 0 \,. \tag{A-7}$$

We then obtain from Eq. A-1,

$$\frac{\partial \rho}{\partial t} + \rho \frac{\partial v}{\partial x} = 0$$

or

$$\frac{\partial v}{\partial x} = - \frac{1}{\rho} \frac{\partial \rho}{\partial t} \,. \tag{A-8}$$

Assuming a constant bore area, A_B, (i.e., no chambrage) and noting that $\rho = C/A_B y$, Eq. A-8 becomes

$$\frac{\partial v}{\partial x} = \frac{1}{y} \frac{\partial y}{\partial t} = \frac{u_p}{y} \,. \tag{A-9}$$

Integrating Eq. A-9 over the region [0,x], we obtain

$$v(x) = \left(\frac{x}{y}\right) u_p \,. \tag{A-10}$$

Corner notes that the term "Lagrange approximation" is applied to Eq. A-7 or A-10 and the Eq. A-7 leads to Equation A-10, but it is not true that Eq. A-10 necessarily implies Eq. A-7 (Corner 1950).

Substituting Eq. A-10 in Eq. A-2, we have

$$\left(\frac{x}{y}\right) \dot{u}_p = -\frac{1}{\rho}\frac{\partial P}{\partial x}$$

where $\dot{u}_p \equiv du_p/dt$, or

$$\frac{\partial P}{\partial x} = -\frac{C\dot{u}_p}{A_B}\left(\frac{x}{y^2}\right). \tag{A-11}$$

Integrating on [x,y], and noting that

$$\dot{u}_p = \frac{(P_{BASE} - P_{RES})\, A_B}{M} \,,$$

$$P(0) = P_{BREECH} \,,$$

and

$$P(y) = P_{BASE} \,,$$

where P_{RES} is the bore resistance pressure and M is the projectile mass, we obtain

$$P(x) = P_{BASE} + \frac{C}{2M}(P_{BASE} - P_{RES})\left(1 - \frac{x^2}{y^2}\right). \quad \text{(A-12)}$$

For $x = 0$,

$$P_{BREECH} = P_{BASE} + \frac{C}{2M}(P_{BASE} - P_{RES}) . \quad \text{(A-13)}$$

The space mean pressure is defined by

$$\bar{P} = \frac{1}{y}\int_0^y P(x)dx , \quad \text{(A-14)}$$

and, upon substituting Eq. A-12, we obtain

$$\bar{P} = P_{BASE} + \frac{C}{3M}(P_{BASE} - P_{RES}) . \quad \text{(A-15)}$$

The kinetic energy associated with the motion of the gas is

$$KE_{GAS} = \int_0^y \frac{1}{2} A_B \rho v^2 dx . \quad \text{(A-16)}$$

Using Eq. A-10, we obtain

$$KE_{GAS} = \frac{1}{2} A_B \rho \frac{u_p^2}{y^2}\left[\frac{1}{3}x^3\right]_0^y$$

or

$$KE_{GAS} = \frac{1}{6} C u_p^2 . \quad \text{(A-17)}$$

APPENDIX B:

THE PIDDUCK-KENT SPECIAL SOLUTION FOR THE MOTION OF THE POWDER GAS IN A GUN

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The Pidduck-Kent gradient model, like the Lagrange model, is an approximate solution of the continuity and momentum equations for the gaseous propelling charge in the Lagrange ballistic problem. This solution was first suggested by Love and Pidduck (1922). Kent (1936) noted an error in Pidduck's solution and presented solutions for both ideal and nonideal gases. Vinti and Kravitz (1949) have published a detailed treatment of the Pidduck-Kent gradient model, along with tables for use in computing the parameters which arise in the model. Corner (1950) also discusses the development of the Pidduck-Kent gradient model

The Lagrange ballistic problem is an idealization of the interior ballistic process in a gun. It is assumed that the gun is a right circular cylinder (i.e., the chamber and bore diameter are equal) closed at one end by the breech. It is also assumed that prior to the start of projectile motion, the propellant is instantaneously burned, such that the chamber is initially filled with a quiescent gas of uniform pressure, density, and temperature.* The gas is assumed to be inviscid and heat loss to the walls is neglected, such that the flow is isentropic.

At time $t = 0$, the initial pressure and temperature distributions (as in the case of the Lagrange gradient model), as well as the initial density distribution of the Pidduck-Kent solution are nonuniform, decreasing monotonically from the breech to the base of the projectile. Fixed gradients (i.e., distributions) of system properties throughout the interior ballistic process are an inherent characteristic of gradient models. These nonequilibrium initial distributions are physically unrealistic, but the spatial averages of these distributions must be consistent with the initial equilibrium conditions of the Lagrange problem. However, as demonstrated computationally in this work, for finite charge-to-mass ratios, the wave dynamic solution appears to approach the special solution for large projectile travel, as has been traditionally assumed.

In this Appendix, a detailed development of the Pidduck-Kent gradient model is presented. This derivation follows closely that of Vinti and Kravitz (1949). However, the notation is modified slightly for clarity.

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* Although the Lagrange and Pidduck-Kent gradient models arise in the solution of an idealized problem, they are routinely applied in the simulation of real gun systems. In such cases, the gas is replaced by a "fluid" consisting of a mixture of the combustion gas and burning solid propellant grains. The density of this fluid is based on the combined masses of the gas and solid phases.

A Lagrangian formulation is used in the development of the Pidduck-Kent gradient model. The gas column between the breech and the projectile base is initially divided along the axial direction into a very large number of thin, disk-shaped elements. A disk may expand (or contract) axially, but the mass of gas in each disk is constant (i.e., there is no flow of gas between adjacent disks). (Eventually, the number of disks will be permitted to become infinite, [i.e., the thickness of the disks will be allowed to approach zero] such that the solution is continuous.) The thermodynamic quantities for the gas (i.e., pressure, density, etc.) are assumed to be uniform within each element, but are assumed to vary from element to element in the axial direction. Let x be defined as the distance of the midpoint of a given gas element from the breech and let y be the position of the projectile base along the barrel with respect to the breech at some arbitrary time, t, as shown in Figure B-1. We define

$$x\,(t = 0) \equiv x_0 \tag{B-1}$$

where the subscript "0" denotes the value of a given quantity at $t = 0$. The gas is assumed to obey the Nobel-Able equation of state, such that for a gas element initially located at x_0

$$P_0\,(x_0,0)\left[\frac{1}{\rho_0(x_0,0)} - \eta\right] = RT_0(x_0,0)\ , \tag{B-2}$$

where $P_0(x_0,0)$ is the pressure of the gas in the element located at x_0 at $t = 0$, $\rho_0(x,0)$ is the gas density, and $T_0(x_0,0)$ is the gas temperature. As is implied by Eq. B-2, the gas covolume, specific heats, impetus, etc. are assumed to be constants for all elements from the breech to the projectile base. As in the development of the Lagrange gradient model, space mean thermodynamic quantities are defined as spatial averages on [0,y] and, for consistency, the initial values of these space mean quantities must correspond to the uniform initial properties of the gas in the Lagrange problem. This will be addressed in more detail later.

As the projectile accelerates down the barrel, each gas element is assumed to expand isentropically,

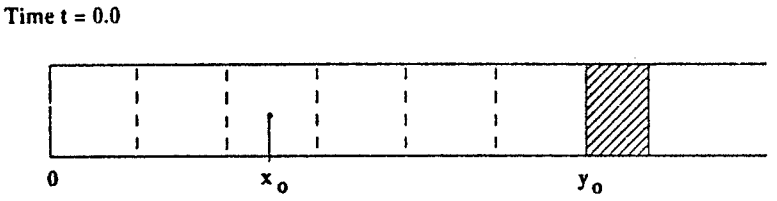

Figure B-1. Gas Column Between the Breech and the Projectile Base.

$$P(x_0,t)\left[\frac{1}{\rho(x_0,t)} - \eta\right]^{\gamma} = P_0(x_0,0)\left[\frac{1}{\rho_0(x_0,0)} - \eta\right]^{\gamma} = K(x_0) \qquad \text{(B-3)}$$

where the adiabatic constant $K(x_0)$ is a function of the initial position of the given gas element. In order to make the problem more tractable, it is assumed that all gas elements follow the same adiabatic, K, such that

$$P(x_0,t)\left[\frac{1}{\rho(x_0,t)} - \eta\right]^{\gamma} = K. \qquad \text{(B-4)}$$

Consider now an element of gas initially located at x_0 with thickness Δx_0 such that the mass of the gas in the element is $\rho(x_0,0)\, A_B\, \Delta x_0$, where A_B is the cross-sectional area of the bore. At some later time, t, the gas element will have moved to a position $x(x_0, t)$ and will have expanded such that its thickness is $\Delta x(x_0,t)$ as shown in Figure B-2. Taking the limit as $\Delta x(x_0,t)$ approaches zero, the continuity equation for this gas element is then

$$\rho_0(x_0,0)\, dx_0 = \rho(x_0,t)\, dx(x_0,t) \qquad \text{(B-5)}$$

and the equation of motion is

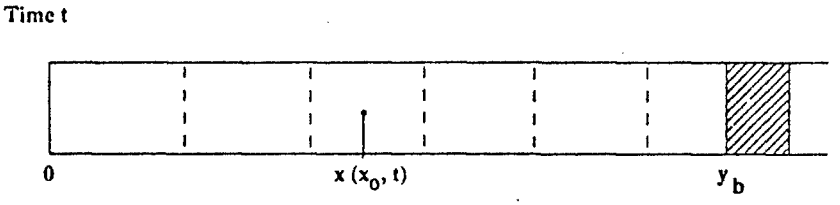

Figure B-2. Evolution of a Gas Element From Time $t = 0$ to Time t.

$$\rho(x_0,t)\ \frac{\partial^2 x(x_0,t)}{\partial t^2} = -\frac{\partial P(x_0,t)}{\partial x}\ , \tag{B-6}$$

which are the continuity and momentum equations in a Lagrangian frame of reference.

Following Vinti and Kravitz (1949), we introduce a new variable

$$z(x_0,t) \equiv x(x_0,t) - \eta \int_0^{x_0} \rho_0(x_0',0)\, dx_0'\ , \tag{B-7}$$

such that

$$\frac{\partial z(x_0,t)}{\partial x_0} = \frac{\partial x(x_0,t)}{\partial x_0} - \eta\rho_0(x_0,0)\ , \tag{B-8}$$

and

$$\frac{z(x_0,t)}{\partial t} = \frac{\partial x(x_0,t)}{\partial t}\ . \tag{B-9}$$

Rewriting the continuity equation (Eq. B-5)

$$\frac{\partial x(x_0,t)}{\partial x_0} = \frac{\rho_0(x_0,0)}{\rho(x_0,t)} \tag{B-10}$$

and substituting in Eq. B-8, we have

$$\frac{\partial z(x_0,t)}{\partial x_0} = \rho_0(x_0,0)\left[\frac{1}{\rho(x_0,t)} - \eta\right]. \tag{B-11}$$

We now define

$$W(x_0,t) = \left[\frac{1}{\rho(x_0,t)} - \eta\right]^{-1} \tag{B-12}$$

such that Eq. B-4 becomes

$$P(x_0,t) = K[W(x_0,t)]^{\gamma} \tag{B-13}$$

and Eq. B-11 becomes

$$\frac{\partial z(x_0,t)}{\partial x_0} = \rho_0(x_0,0)[W(x_0,t)]^{-1} . \tag{B-14}$$

Using Eq. B-9, B-10, and B-13, the equation of motion can be rewritten as

$$\frac{\partial^2 z(x_0,t)}{\partial t^2} = - \frac{K}{\rho_0(x_0,0)} \frac{\partial[W(x_0,t)]^{\gamma}}{\partial x_0} , \tag{B-15}$$

where we have made use of

$$\frac{\partial P(x_0,t)}{\partial x} = \frac{\partial P(x_0,t)}{\partial x_0}\frac{\partial x_0}{\partial x} = \frac{\rho(x_0,t)}{\rho_o(x_0,0)}\frac{\partial P(x_0,t)}{\partial x_0} . \tag{B-16}$$

We now attempt a solution of Eq. B-15 by separation of variables assuming a solution of the form

$$z(x_0,t) = f(x_0)\Phi(t) . \tag{B-17}$$

At $t = 0$,

$$z(x_0,0) = f(x_0)\Phi(0) = \left[x_0 - \eta\int_0^{x_0}\rho_0(x_0',0)\,dx_0'\right]\Phi(0) \tag{B-18}$$

such that

$$f(x_0) = x_0 - \eta\int_0^{x_0}\rho_0(x_0',0)\,dx_0' \tag{B-19}$$

and

$$\Phi(0) = 1 . \tag{B-20}$$

Substituting in Eq. B-14, we obtain

$$\frac{\partial z(x_0,t)}{\partial x_0} = \frac{df(x_0)}{dx_0}\Phi(t) = \rho_0(x_0,0)[W(x_0,t)]^{-1}$$

and using Eq. B-19 to evaluate $\frac{df(x_0)}{dx_0}$ we have

$$[1 - \eta \rho_0(x_0, 0)]\Phi(t) = \rho_0(x_0,0)[W(x_0,t)]^{-1}$$

or, using Eq. B-12 with $W_0(x_0, 0) \equiv W(x_0, 0)$ to emphasize the initial condition,

$$\left[W(x_0,t)\right]^{-1} = \left[W_0(x_0,0)\right]^{-1}\Phi(t) \; . \tag{B-21}$$

Using Eq. B-17 and B-21 in Eq. B-15, we have

$$f(x_0)\frac{d^2\Phi(t)}{dt^2} = -\frac{K}{\rho_0(x_0,0)}[\Phi(t)]^{-\gamma}\frac{\partial[W_0(x_0,0)]^{\gamma}}{\partial x_0} \tag{B-22}$$

and separating variables, we obtain

$$[\Phi(t)]^{\gamma}\frac{d^2\Phi(t)}{dt^2} = B = -\frac{K}{f(x_0)\rho_0(x_0,0)}\frac{\partial[W_0(x_0,0)]^{\gamma}}{\partial x_o} \; , \tag{B-23}$$

where B is the constant of separation. Consider now

$$\frac{-\gamma K[W_0(x_0,0)]^{\gamma-1}}{f(x_0)\rho_0(x_0,0)}\frac{\partial W_0(x_0,0)}{\partial x_0} = B \; , \tag{B-24}$$

where

$$\frac{\partial W_0(x_0,0)}{\partial x_0} = \frac{\partial W_0(x_0,0)}{\partial f(x_0)}\frac{df(x_0)}{dx_0} = \frac{\partial W_0(x_0,0)}{\partial f(x_0)}[1 - \eta\rho_0(x_0,0)] \; , \tag{B-25}$$

such that

$$\frac{-\gamma K[W_0(x_0,0)]^{\gamma-2}}{f(x_0)}\frac{\partial W_0(x_0,0)}{\partial f(x_0)} = B$$

or

$$[W_0(x_0,0)]^{\gamma-2}\, dW_0(x_0,0) = -\frac{B}{\gamma K} f(x_0)\, df(x_0)\ . \tag{B-26}$$

Noting that $f(0) = 0$ and integrating, we obtain

$$[W_0(x_0,0)]^{\gamma-1} - [W_0(0,0)]^{\gamma-1} = \frac{-B(\gamma-1)}{2K\gamma}[f(x_0)]^2,$$

or

$$W_0(x_0,0) = W_0(0,0)\left[1 - af^2(x_0)\right]^{\frac{1}{\gamma-1}} \tag{B-27}$$

where

$$a \equiv \frac{B(\gamma-1)}{2K\gamma[W_0(0,0)]^{\gamma-1}}\ . \tag{B-28}$$

The total mass of propelling gas charge in the chamber, C, is given by

$$C \equiv A_B \int_0^{y_0} \rho_0(x_0',0)\, dx_0' \tag{B-29}$$

where y_0 is the initial position of the pose of the projectile. Rewriting Eq. B-29 in terms of $f(x_0)$, we have

$$C = A_B \int_0^{f(y_0)} \rho_0(x_0',0) \frac{dx_0'}{df(x_0')}\, df(x_0') = A_B \int_0^{f(y_0)} \frac{\rho_0(x_0',0)}{1 - \eta\rho_0(x_0',0)}\, df(x_0'), \tag{B-30}$$

where we have again used Eq. B-19 to evaluate $df(x_0)/dx_0$. Noting that

$$\frac{\rho_0(x_0,0)}{1 - \eta\rho_0(x_0,0)} = \left[\frac{1}{\rho_0(x_0,0)} - \eta\right]^{-1} \equiv W_0(x_0,0)\ ,$$

defining

$$f_b = f(y_0) = y_0 - \eta C/A_B \tag{B-31}$$

and using Eq. B-27, we obtain

$$C = A_B W_0(0,0) \int_0^{f_b} \left[1 - a f^2(x_0')\right]^{\frac{1}{\gamma-1}} df(x_0') \,. \tag{B-32}$$

Defining

$$a_0 \equiv a f_b^2 \tag{B-33}$$

and

$$\mu(x_0) \equiv \frac{f(x_0)}{f_b} \,, \tag{B-34}$$

such that

$$\mu(0) = 0, \tag{B-35}$$

$$\mu(y_0) = 1 \,, \tag{B-36}$$

and

$$df(x_0) = f_b \, d\mu(x_0) \,, \tag{B-37}$$

Eq. B-32 becomes

$$C = A_B f_b W_0(0,0) \int_0^1 \left(1 - a_0 \mu^2(x_0)\right)^n d\mu(x_0) \,, \tag{B-38}$$

where $n \equiv \frac{1}{\gamma - 1}$ is the polytropic index. Vinti and Kravitz (1949) have written Eq. B-38 as

$$C = A_B f_b W_0(0,0) \; S(n, a_0) \tag{B-39}$$

where

$$S(n, a_0) \equiv \int_0^1 \left(1 - a_0 \mu^2(x_0)\right)^n d\mu(x_0). \tag{B-40}$$

It is convenient to rewrite Eq. B-40 at this point. Let

$$q(\mu) \equiv a_0 \mu^2(x_0)$$

such that

$$d\mu(x_0) = \frac{dq(\mu)}{2\sqrt{a_0 q(\mu)}} .$$

Substituting in Eq. B-40, we obtain

$$S(n, a_0) = \frac{1}{2\sqrt{a_0}} \int_0^{a_0} q(\mu)^{-\frac{1}{2}} (1 - q(\mu))^n dq(\mu) \tag{B-41}$$

which gives us

$$S(n, a_0) = \frac{1}{2\sqrt{a_0}} B_{a_0}\left(\frac{1}{2}, n + 1\right) = {}_2F_1\left(-n, \frac{1}{2}; \frac{3}{2}; a_0\right) \tag{B-42}$$

where $B_{a_0}(\frac{1}{2}, n + 1)$ is the incomplete beta function and ${}_2F_1(-n, \frac{1}{2}; \frac{3}{2}; a_0)$ is Gauss' hypergeometric function. Eq. B-42 will prove useful later in our development of the Pidduck-Kent gradient model.

Eq. B-40 could be utilized to calculate the parameter a_0 given a detailed description of the initial state of the system. However, Vinti and Kravitz (1949) have developed a more general equation from which a_0 can be determined. Note that at the base of the projectile, the equation of motion becomes

$$\frac{\partial^2 z(y_0,t)}{\partial t^2} = \frac{P_b A_B}{M} ,$$

where M is the projectile mass and P_b is the base pressure, or, using Eq. B-17,

$$f_b \frac{d^2\Phi(t)}{dt^2} = \frac{P_b A_B}{M} , \tag{B-43}$$

where $P(y_0, t) \equiv P_b$. Using Eq. B-13 and B-21, we have

$$P_b = K [W_0(y_0,0)]^\gamma [\Phi(t)]^{-\gamma}$$

such that

$$f_b \frac{d^2\Phi(t)}{dt^2} = \frac{A_B K}{M} [W_0(y_0,0)]^\gamma [\Phi(t)]^{-\gamma} . \tag{B-44}$$

Separating variables we obtain

$$[\Phi(t)]^\gamma \frac{d^2\Phi(t)}{dt^2} = B = \frac{A_B K}{M f_b} [W_0(y_0,0)]^\gamma \tag{B-45}$$

where B is the constant of separation introduced in Eq. B-23. Using Eq. B-28 and B-33, we have

$$\frac{2 K a_0 \gamma}{(\gamma - 1) f_b^2} [W_0(0,0)]^{\gamma-1} = B = \frac{A_B K}{M f_b} [W_0(y_0,0)]^\gamma$$

or

$$\frac{2\gamma a_0 M}{\gamma - 1} = A_B f_b W_0(0,0) \left[\frac{W_0(y_0,0)}{W_0(0,0)} \right]^\gamma$$

which, with the use of Eq. B-27 and B-33, becomes

$$\frac{2\gamma a_0 M}{\gamma - 1} = A_B f_b W_0(0,0)\left[1 - a_0\right]^{\frac{\gamma}{\gamma-1}} . \qquad \text{(B-46)}$$

Multiplying both sides of Eq. B-46 by $S(n,a_0)$ and using Eq. B-39, we obtain

$$\frac{2\gamma}{\gamma - 1} a_0 (1 - a_0)^{-\frac{\gamma}{\gamma-1}} S(n,a_0) = \frac{C}{M} \equiv \varepsilon , \qquad \text{(B-47)}$$

which can be used to calculate the parameter a_0 given only the ratio of specific heats of the gas and the propelling charge to projectile mass ratio, C/M.

Ratio of Breech to Base Pressure

The determination of the ratio of breech pressure to projectile base pressure is now straightforward. Using Eq. B-27 and B-33, we have

$$W_0(y_c,0) = W_0(0,0)\left[1 - a_0\right]^{\frac{1}{\gamma-1}} \qquad \text{(B-48)}$$

or, dividing both sides by $\Phi(t)$ and using Eq. B-21,

$$W(y_0,t) = W(0,t)\left[1 - a_0\right]^{\frac{1}{\gamma-1}} . \qquad \text{(B-49)}$$

We now make use of Eq. B-13 to obtain

$$\frac{P(0,t)}{P(y_0,t)} = \left[1 - a_0\right]^{\frac{-\gamma}{\gamma-1}} = \left[1 - a_0\right]^{-(n+1)} \qquad \text{(B-50)}$$

which demonstrates that at all times during gas expansion the ratio of breech to projectile base pressure is a constant dependent on γ and a_0.

Space Mean Pressure, Density, and Temperature

As noted earlier, space mean thermodynamic quantities are defined as spatial averages on [0,y] and, for consistency, the initial values of these quantities must correspond to the uniform initial state of the gas in the Lagrange problem. As a first step, we write the equations for the spatial distributions of pressure, temperature, and density in the gas. From Eq. B-12, B-13, B-21, B-27, B-33, and B-34, we have

$$\left[\frac{1}{\rho(x_0,t)} - \eta\right] = \left[\frac{1}{\rho_0(0,0)} - \eta\right]\left[1 - a_0\mu^2(x_0)\right]^{-n}\Phi(t), \tag{B-51}$$

$$P(x_0,t) = P_0(0,0)\left[1 - a_0\mu^2(x_0)\right]^{n+1}[\Phi(t)]^{-\gamma} \tag{B-52}$$

and,

$$T(x_0,t) = T_0(0,0)\left[1 - a_0\mu^2(x_0)\right][\Phi(t)]^{1-\gamma}. \tag{B-53}$$

Consider first the gas density. We rewrite Eq. B-51 in the form

$$W(x_0,t) = W_0(0,0)\left[1 - a_0\mu^2(x_0)\right]^{n}\Phi(t)^{-1}. \tag{B-54}$$

Performing a spatial average over $0 \le \mu \le 1$, we obtain

$$\overline{W(t)} = W_0(0,0)[\Phi(t)]^{-1}\int_0^1\left[1 - a_0\mu^2(x_0)\right]^{n}d\mu(x_0)$$

or, using Eq. B-40,

$$\overline{W(t)} = W_0(0,0)[\Phi(t)]^{-1}\,{}_2F_1\left(-n, \frac{1}{2}; \frac{3}{2}; a_0\right). \tag{B-55}$$

This result can also be obtained from Eq. B-7 and B-17,

$$z(y_0,t) = f_b\Phi(t) = y(t) - \eta\int_0^{y_0}\rho_0\left(x_0',0\right)dx_0' = y(t) - C\eta/A_B \qquad \text{(B-56)}$$

where we have noted that $x(y_0, t) \equiv y(t)$ is the position of the projectile base as a function of time. Since the average gas density, $\overline{\rho(t)}$, is the gas mass divided by the gas volume,

$$f_b\Phi(t) = \frac{C}{A_B}\left(\frac{1}{\overline{\rho(t)}} - \eta\right), \qquad \text{(B-57)}$$

where *y(t)* is the position of the projectile base at some time, *t*, after the start of projectile motion. Rearranging and using Eq. B-39 and B-42, we have

$$\left(\frac{1}{\overline{\rho(t)}} - \eta\right) = \frac{A_B f_b}{C}\Phi(t) \qquad \text{(B-58)}$$

or,

$$\overline{W(t)} = W_0(0,0)[\Phi(t)]^{-1}\ {}_2F_1\left(-n,\frac{1}{2};\frac{3}{2},a_0\right). \qquad \text{(B-59)}$$

If we approach this problem in a slightly different manner using Eq. B-17 and B-31, we obtain

$$z(y_0,t) = f_b\Phi(t) = (y_0 - \eta C/A_B)\Phi(t)$$

or,

$$f_b\Phi(t) = \frac{C}{A_B}\left(\frac{1}{\overline{\rho(0)}} - \eta\right)\Phi(t).$$

Again rearranging and using Eq. B-39 and B-42, we have

$$\left(\frac{1}{\overline{\rho(0)}} - \eta\right)\Phi(t) = \frac{A_B f_b}{C}\Phi(t)$$

or, using Eq. B-12

$$\overline{W(0)}[\Phi(t)]^{-1} = W_0(C,0)[\Phi(t)]^{-1}\,{}_2F_1\left(-n,\frac{1}{2};\frac{3}{2};a_0\right), \tag{B-60}$$

which, with Eq. B-59, gives us

$$[\overline{W(t)}]^{-1} = [\overline{W(0)}]^{-1}\Phi(t)\ . \tag{B-61}$$

Thus, the spatial average of $\overline{W(t)}$ which is a function of the gas density, has the same time dependence as $W(x_0, t)$ for each individual gas element, which is given by Eq. B-21. This result might well have been anticipated as a result of the assumption that all gas elements follow the same adiabatic.

We now consider the spatial average of the gas temperature, which will be defined in terms of the internal energy of the gas, i.e.,

$$E_i(t) \equiv C\ \ c_v\,\overline{T(t)} \tag{B-62}$$

where $E_i(t)$ is the internal energy of the gas at any time, t, and c_v is the specific heat at constant volume. Since energy, like mass, is an extensive quantity, we have, using Eq. B-29,

$$E_i(t) = A_B c_v \int_0^{y(t)} T(x_0,t)\,\rho(x_0,t)\,dx(x_0,t) \tag{B-63}$$

where the integration is now over the gas column at some arbitrary time, t. We first use Eq. B-7 to change variables

$$z(x_0,t) = x(x_0,t) - \eta\int_0^{x_0}\rho\left(x_0',0\right)dx_0'\ ,$$

where, from Eq. B-56,

$$z_b \equiv z(y_0, t) = y(t) - \eta C / A_b \, .$$

Using Eq. B-10 in B-11, we have

$$dx(x_0, t) = \left[1 - \eta \rho(x_0, t)\right]^{-1} dz(x_0, t) \tag{B-64}$$

such that Eq. B-63 becomes

$$E_I(t) = A_B c_v \int_0^{z_b} T(x_0, t) \left[\frac{1}{\rho(x_0, t)} - \eta\right]^{-1} dz(x_0, t) \, . \tag{B-65}$$

We now define a new variable

$$\mu(x_0) \equiv \frac{z(x_0, t)}{z(y_0, t)} = \frac{z(x_0, t)}{z_b} = \frac{f(x_0)\Phi(t)}{f(y_0)\Phi(t)} = \frac{f(x_0)}{f_b} \tag{B-66}$$

which is identical to Eq. B-34, our original definition of $\mu(x_0)$. We now note that

$$dz(x_0, t) = z_b \, d\mu(x_0) = f_b \Phi(t) \, d\mu(x_0) \tag{B-67}$$

where Eq. B-63 and Eq. B-65 are taken at a fixed time, t, and thus

$$\mu(0) = 0,$$

and

$$\mu(y_0) = 1$$

as before. Using Eq. B-51, B-53, and B-67 in Eq. B-65, we obtain

$$E_i(t) = A_B c_v T_0(0,0)\left[\frac{1}{\rho_0(0,0)} - \eta\right]^{-1} [\Phi(t)]^{-\gamma} z_b \int_0^1 \left[1 - a_0 \mu^2(x_0)\right]^{n+1} d\mu(x_0) \quad \text{(B-68)}$$

or, using Eq. B-17, B-31, and B-40,

$$E_i(t) = c_v T_0(0,0)\left[\frac{1}{\rho_0(0,0)} - \eta\right]^{-1} A_B f_b \; {}_2F_1\left(-n-1, \frac{1}{2}; \frac{3}{2}; a_0\right)[\Phi(t)]^{1-\gamma}.$$

Using Eq. B-39 and B-42, we note that

$$A_B f_b\left[\frac{1}{\rho_0(0,0)} - \eta\right]^{-1} = \frac{C}{{}_2F_1\left(-n, \frac{1}{2}; \frac{3}{2}; a_0\right)} \quad \text{(B-69)}$$

such that Eq. B-68 becomes

$$E_i(t) = C c_v T_0(0,0)[\Phi(t)]^{1-\gamma} \frac{{}_2F_1\left(-n-1, \frac{1}{2}; \frac{3}{2}; a_0\right)}{{}_2F_1\left(-n, \frac{1}{2}; \frac{3}{2}; a_0\right)} \quad \text{(B-70)}$$

and, using Eq. B-62, we obtain the space mean temperature

$$\overline{T(t)} = T_0(0,0)\,[\Phi(t)]^{1-\gamma} \frac{{}_2F_1\left(-n-1, \frac{1}{2}; \frac{3}{2}; a_0\right)}{{}_2F_1\left(-n, \frac{1}{2}; \frac{3}{2}; a_0\right)} \quad \text{(B-71)}$$

or, using Eq. B-53, and noting that $\mu(0) = 0$,

$$\overline{T(t)} = T(0,t)\ \frac{{}_2F_1\left(-n-1,\frac{1}{2};\frac{3}{2};a_0\right)}{{}_2F_1\left(-n,\frac{1}{2};\frac{3}{2};a_0\right)}\ . \tag{B-72}$$

In order to determine the space mean pressure, we note that

$$\overline{P(t)}\ \left[\frac{1}{\overline{\rho(t)}} - \eta\right] = R\ \overline{T(t)} \tag{B-73}$$

or, using Eq. B-12,

$$\overline{P(t)} = R\ \overline{T(t)}\ \overline{W(t)}\ . \tag{B-74}$$

Combining Eq. B-55 and B-72 and using Eq. B-21, we obtain

$$\overline{P(t)} = RT(0,t)\ \ W(0,t)\ \ {}_2F_1\left(-n-1,\frac{1}{2};\frac{3}{2};a_0\right)$$

or

$$\overline{P(t)} = P(0,t)\ \ {}_2F_1\left(-n-1,\frac{1}{2};\frac{3}{2};a_0\right)\ . \tag{B-75}$$

Now we divide Eq. B-75 by $P(y_0,t)$ and use Eq. B-50 to obtain

$$\frac{\overline{P(t)}}{P(y_0,t)} = (1-a_0)^{-(n+1)}\ {}_2F_1\left(-n-1,\frac{1}{2};\frac{3}{2};a_0\right)\ . \tag{B-76}$$

From Eq. B-47, we have

$$(1-a_0)^{-(n+1)} = \frac{\varepsilon}{2(n+1)\,a_0\ {}_2F_1\left(-n,\frac{1}{2};\frac{3}{2};a_0\right)} \tag{B-77}$$

which in Eq. B-76 gives us

$$\frac{\overline{P(t)}}{P(y_0,t)} = \frac{\varepsilon}{2(n+1)a_0} \frac{{}_2F_1(-n-1,\frac{1}{2};\frac{3}{2};a_0)}{{}_2F_1(-n,\frac{1}{2};\frac{3}{2};a_0)}. \tag{B-78}$$

It can be shown that (see Appendix C)

$${}_2F_1(-n-1,\frac{1}{2};\frac{3}{2};a_0) = (1-a_0)^{n+1} + \frac{2a_0(n+1)}{3}\ {}_2F_1(-n,\frac{3}{2};\frac{5}{2};a_0) \tag{B-79}$$

or, using Eq. B-77

$$\begin{aligned} {}_2F_1(-n-1,\frac{1}{2};\frac{3}{2};a_0) = \frac{2(n+1)a_0}{\varepsilon}\ {}_2F_1(-n,\frac{1}{2};\frac{3}{2};a_0) \\ + \frac{2a_0(n+1)}{3}\ {}_2F_1(-n,\frac{3}{2};\frac{5}{2};a_0) \end{aligned} \tag{B-80}$$

which, in Eq. B-78, gives us

$$\frac{\overline{P(t)}}{P(y_0,t)} = 1 + \frac{\varepsilon}{3} \frac{{}_2F_1(-n,\frac{3}{2};\frac{5}{2};a_0)}{{}_2F_1(-n,\frac{1}{2};\frac{3}{2};a_0)}. \tag{B-81}$$

or

$$\frac{\overline{P(t)}}{P(y_0,t)} = 1 + \frac{\varepsilon}{\delta} \tag{B-82}$$

where

$$\frac{1}{\delta} \equiv \frac{1}{3} \frac{{}_2F_1(-n,\frac{3}{2};\frac{5}{2};a_0)}{{}_2F_1(-n,\frac{1}{2};\frac{3}{2};a_0)}. \tag{B-83}$$

Noting that

$$ {}_2F_1\left(-n, \frac{1}{2}; \frac{3}{2}; a_0\right) $$

is real and finite on $0 \le a_0 \le 1$ for $n > 1$ of practical interest, we see from Eq. B-47 that $a_0 = 0$ corresponds to $\varepsilon = 0$ and that $a_0 = 1$ corresponds to $\varepsilon \to \infty$. Using the definition of the Gauss' hypergeometric function (see Appendix C) we see from Eq. B-83 that $\delta = 3$ for $\varepsilon = a_0 = 0$ (as expected) and $\delta = 2n + 3$ for $a_0 = 1$. Thus, in the limit of zero C/M the Pidduck-Kent solution reduces to that of Lagrange.

As is also demonstrated in Appendix C, a relationship can be derived among the parameters a_0, ε, n, and δ,

$$ \frac{1}{\delta} = \frac{1}{2n + 3}\left[\frac{1}{a_0} - \frac{2(n + 1)}{\varepsilon}\right] \qquad \text{(B-84)} $$

which can be used to determine δ for a given ε and n once a_0 has been calculated. For small ε, it is easily shown from Eq. B-84 that a_0 approaches $\varepsilon/2(n + 1)$.

Projectile Velocity

The velocity of an element of gas is given by Eq. B-9

$$ \frac{\partial x(x_0, t)}{\partial t} = \frac{\partial z(x_0, t)}{\partial t} $$

and, using Eq. B-17, we obtain

$$ u(x_0, t) = \frac{\partial x(x_0, t)}{\partial t} = f(x_0)\,\frac{d\Phi(t)}{dt} \qquad \text{(B-85)} $$

from Eq. B-23, we have

$$ \frac{d^2\Phi(t)}{dt^2} = B[\Phi(t)]^{-\gamma}. \qquad \text{(B-86)} $$

We now define

$$\Psi(t) \equiv \frac{d\Phi(t)}{dt}$$

such that Eq. B-86 can be rewritten in the form

$$\frac{d^2\Phi(t)}{dt^2} = \frac{d\Psi(t)}{d\Phi(t)}\frac{d\Phi(t)}{dt} = \Psi(t)\frac{d\Psi(t)}{d\Phi(t)} = B[\Phi(t)]^{-\gamma}$$

which, upon integrating with respect to $\Phi(t)$ and noting that $\Phi(0) = 1$, gives us

$$[\Psi(t)]^2 = \left(\frac{d\Phi(t)}{dt}\right)^2 = \frac{2B}{\gamma - 1}\left\{1 - [\Phi(t)]^{-(\gamma-1)}\right\} \tag{B-87}$$

and from Eq. B-85

$$u^2(x_0,t) = [f(x_0)]^2\frac{2B}{\gamma - 1}\left\{1 - [\Phi(t)]^{-(\gamma-1)}\right\}. \tag{B-88}$$

From Eq. B-28 and B-33, we have

$$B = \frac{2\gamma K a_0}{(\gamma - 1)f_b^2}[W_0(0,0)]^{\gamma-1} \tag{B-89}$$

where, from Eq. B-13,

$$K \equiv P_0(0,0)[W_0(0,0)]^{-\gamma}$$

such that

$$B = \frac{2\gamma a_0}{(\gamma - 1)f_b^2}P_0(0,0)[W_0(0,0)]^{-1}$$

or

$$B = \frac{2\gamma a_0}{(\gamma - 1)f_b^2}RT_0(0,0) \tag{B-90}$$

which gives us

$$u^2(x_0,t) = \frac{4\gamma RT_0(0,0)}{(\gamma-1)^2}\, a_0\mu^2(x_0)\left\{1 - [\Phi(t)]^{-(\gamma-1)}\right\}. \tag{B-91}$$

At the projectile base, $\mu(y_0) \equiv 1$, such that

$$u^2(y_0,t) = \frac{4\gamma RT_0(0,0)}{(\gamma-1)^2}\, a_0\left\{1 - [\Phi(t)]^{-(\gamma-1)}\right\}. \tag{B-92}$$

If we now use Eq. B-61 to obtain an expression for $\Phi(t)$, we have

$$\Phi(t) = \frac{\overline{W(0)}}{\overline{W(t)}} = \left[\frac{\dfrac{1}{\overline{\rho(t)}} - \eta}{\dfrac{1}{\overline{\rho(0)}} - \eta}\right] = \left[\frac{y(t) - \eta C/A_B}{y_0 - \eta C/A_B}\right] \tag{B-93}$$

and defining the expansion ratio* of the gas as

$$\varepsilon_f \equiv \frac{y(t) - \eta C/A_B}{y_0 - \eta C/A_B} = \Phi(t) \tag{B-94}$$

we have for the projectile velocity

$$u^2(y_0,t) = \frac{4\gamma RT_0(0,0)}{(\gamma-1)^2}\, a_0\left[1 - \left(\frac{1}{\varepsilon_f}\right)^{\gamma-1}\right] \tag{B-95}$$

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* Expansion ratio is usually defined in terms of initial chamber length, ℓ_0, and projectile travel, x_p, such that,

$$\varepsilon_f = \frac{x_p + \ell_0 - \eta C/A_B}{\ell_0 - \eta C/A_B} = \frac{x_p + \ell_0(1-\eta\rho_0)}{\ell_0(1-\eta\rho_0)}.$$

in the form presented by Siegel (1979). We now use Eq. B-72 to evaluate T(0,0) in terms of the initial energy of the gas. For $t = 0$, we have

$$\overline{T(0)} = T_0(0,0) \frac{{}_2F_1\left(-n-1, \frac{1}{2}; \frac{3}{2}; a_0\right)}{{}_2F_1\left(-n, \frac{1}{2}; \frac{3}{2}; a_0\right)} \tag{B-96}$$

which, using Eq. B-78 and B-80, gives us

$$\overline{T(0)} = T_0(0,0) \left[\frac{2(n+1)\,a_0}{\varepsilon}\right]\left(1 + \frac{\varepsilon}{\delta}\right)$$

or

$$T_0(0,0) = \overline{T(0)} \left(\frac{\varepsilon}{1 + \frac{\varepsilon}{\delta}}\right) \frac{1}{2(n+1)\,a_0} . \tag{B-97}$$

Substituting Eq. B-97 in Eq. B-95 and noting that

$$\frac{R\overline{T(0)}}{\gamma - 1} = E_i(0) \equiv E_0 , \tag{B-98}$$

the initial specific internal energy of the gas, and

$$\frac{2\gamma}{\gamma - 1} = 2(n+1) ,$$

we obtain

$$u^2(y_0, t) = \frac{2E_0\varepsilon}{1 + \frac{\varepsilon}{\delta}} \left[1 - \left(\frac{1}{\varepsilon_f}\right)^{\gamma-1}\right] \tag{B-99}$$

which is solution to the preburned propellant, nonideal gas gun assuming the Pidduck-Kent gradient model, which is discussed in Appendix E. In the limit of very large projectile travel, Eq. B-95 becomes

$$u_p^2 \approx \frac{4\gamma R T_0(0,0)\, a_0}{(\gamma - 1)^2} \tag{B-100}$$

which, using Eq. B-97 and B-98, gives us

$$u_p^2 \approx \frac{4\gamma R \overline{T(0)}}{(\gamma - 1)^2} \left(\frac{\varepsilon}{\varepsilon + \delta} \right) \frac{\delta}{2(n + 1)} . \tag{B-101}$$

It has been generally supposed that the Pidduck-Kent solution of the Lagrange problem approaches the wave dynamic solution at sufficiently large projectile travel. Corner (1950) has developed an expression of the form of Eq. B-100 for the maximum possible velocity for the Pidduck-Kent gradient model for the case of infinite C/M. Recalling that $a_0 = 1$ for $\varepsilon \rightarrow \infty$, we have

$$u_{max} = \frac{2}{\gamma - 1} \left(\gamma R T_0(0,0) \right)^{\frac{1}{2}} , \tag{B-102}$$

which is identical in form to the escape velocity obtained in the characteristic solution of the Lagrange problem. However, in the Lagrange problem the initial gas pressure, temperature and density are uniform throughout the chamber. Thus, in the initial, physical state of the gas, the temperature at the breech and the space mean temperature are equivalent. In the Pidduck-Kent solution, the initial pressure, density, and temperature distributions decrease monotonically from the breech to the base of the projectile, a physically unrealistic initial state for the system. Thus, the space mean temperature, from which the specific internal energy of the gas is determined, is lower than the initial breech temperature. Therefore, for equivalent gas internal specific energies, the Pidduck-Kent solution results in a maximum (escape) velocity greater than that obtained from the characteristic solution of the Lagrange problem for infinite C/M and infinite travel. From Eq. B-101 we obtain

$$u_{max} = \frac{2}{\gamma - 1} (\gamma R \overline{T(0)})^{\frac{1}{2}} \left(\frac{2n + 3}{2(n + 1)} \right)^{\frac{1}{2}} \qquad \text{(B-103)}$$

or

$$u_{max}^{P-K} = [\frac{3\gamma - 1}{2\gamma}]^{\frac{1}{2}} u_{escape} \, . \qquad \text{(B-104)}$$

Corner (1950) has mistakenly equated the breech temperature in Eq. B-102 with the initial temperature of the Lagrange problem and then concluded that, at least for infinite C/M, the Pidduck-Kent and wave dynamic solutions yield the same maximum velocity. As demonstrated here, the Pidduck-Kent maximum velocity is actually somewhat higher than the escape velocity for $\gamma > 1$. This difference is about 8% for $\gamma = 1.2$, 14% for $\gamma = 1.4$, and 20% for $\gamma = 1.667$. It would appear that, at least for the infinite C/M case, the Pidduck-Kent solution does not approach the wave dynamic solution at large projectile travel, as has been traditionally assumed. However, as shown in the comparisons of calculated trajectories for finite C/M, the Pidduck-Kent solution does appear to approach the wave dynamic solution at sufficiently large projectile travel.

Gas Kinetic Energy

The kinetic energy of the propelling gas at an arbitrary time, t, is defined by

$$KE_{GAS} \equiv \frac{1}{2} \int_0^{y(t)} \rho(x_0, t) A_B \left[\frac{\partial x(x_0, t)}{\partial t} \right]^2 dx(x_0, t) \, . \qquad \text{(B-105)}$$

Using Eq. B-64, B-66, B-67, and B-85 we obtain

$$KE_{GAS} = \frac{1}{2} A_B f_b \Phi(t) \int_0^1 u^2(x_0, t) \left[\frac{1}{\rho(x_0, t)} - \eta \right] d\mu(x_0) \, . \qquad \text{(B-106)}$$

Now, using Eq. B-12, B-51 and B-85, we have

$$KE_{GAS} = \frac{1}{2} A_B f_b \Phi(t) \left[f_b \frac{d\Phi(t)}{dt} \right]^2 W_0(0,0) [\Phi(t)^{-1}] \int_0^1 \left[1 - a_0 \mu^2(x_0) \right]^n \mu^2(x_0) d\mu(x_0), \quad \text{(B-107)}$$

or

$$KE_{GAS} = \frac{1}{6} A_B f_b W_0(0,0) \left[f_b \frac{d\Phi(t)}{dt} \right]^2 {}_2F_1\left(-n, \frac{3}{2}; \frac{5}{2}; a_0\right) . \quad \text{(B-108)}$$

Noting that

$$f_b \frac{d\Phi(t)}{dt} \equiv u(y_0, t)$$

and from Eq. B-39

$$A_B f_b W_0(0,0) = \frac{C}{{}_2F_1\left(-n, \frac{1}{2}; \frac{3}{2}; a_0\right)}$$

we have

$$KE_{GAS} = \frac{1}{6} C u^2(y_0, t) \frac{{}_2F_1\left(-n, \frac{3}{2}; \frac{5}{2}; a_0\right)}{{}_2F_1\left(-n, \frac{1}{2}; \frac{3}{2}; a_0\right)} \quad \text{(B-109)}$$

which, using Eq. B-83, gives us

$$KE_{GAS} = \left(\frac{\varepsilon}{\delta} \right) KE_{PROJ} \quad \text{(B-110)}$$

and

$$KE_{TOTAL} = \left(1 + \frac{\varepsilon}{\delta} \right) KE_{PROJ} . \quad \text{(B-111)}$$

Evaluation of the Function $f(x_0)$:

The equations developed thus far are adequate for solution for the interior ballistic problem to determine projectile velocity as a function of travel. In addition, the breech, projectile base, and space mean values of the thermodynamic functions (i.e., pressure, temperature, etc.) can be calculated as functions of projectile travel. Given the values of these functions at the breech prior to the start of projectile motion, Eq. B-51–B-53 could then be used to evaluate the thermodynamic functions at any arbitrary position along the barrel and, thus, the spatial distribution of the functions, for any given projectile travel. However, Eq. B-51–B-53 express the thermodynamic functions in terms of a dimensionless variable $\mu(x_0)$ and it is, therefore, necessary to determine the relationship between x_0 and $\mu(x_0)$ if we wish to investigate the behavior of spatial variation of these functions in the Pidduck-Kent solution.

The dimensionless function $\mu(x_0)$ is defined by Eq. B-34,

$$\mu(x_0) \equiv \frac{f(x_0)}{f_b} , \tag{B-34}$$

where

$$f(x_0) = x_0 - \eta \int_0^{x_0} \rho_0(x_0',0)\, dx_0' \tag{B-19}$$

and

$$f_b \equiv f(y_0) = y_0 - \eta C/A_B \tag{B-31}$$

such that

$$\mu(x_0) = \frac{1}{f_b}\left[x_0 - \eta \int_0^{x_0} \rho_0(x_0',0)\, dx_0' \right] . \tag{B-112}$$

We must now evaluate the integral, $C(x_0)$,

$$C(x_0) \equiv A_B \int_0^{x_0} \rho(x_0',0)\, dx_0' \tag{B-113}$$

such that

$$\mu(x_0) = \left[\frac{x_0 - \eta\, C(x_0)/A_B}{y_0 - \eta\, C/A_B} \right], \tag{B-114}$$

where $C(x_0)$ is simply the mass of propelling gas between the breech and x_0. We note that $\mu(0) = 0$ and $\mu(y_0) = 1$, as before. The integral $C(x_0)$ can now be evaluated in a manner analogous to the integral in Eq. B-29. Recall that

$$\frac{df(x_0)}{dx_0} = 1 - \eta\, \rho_0(x_0, 0)$$

such that

$$C(x_0) = A_B \int_0^{f(x_0)} \rho_0(x_0', 0) \frac{dx_0'}{df(x_0')}\, df(x_0') = A_B \int_0^{f(x_0)} W_0(x_0', 0)\, df(x_0') .$$

Using Eq. B-27, we obtain

$$C(x_0) = A_B\, W_0(0,0) \int_0^{f(x_0)} \left[1 - a f^2(x_0') \right]^{\frac{1}{\gamma - 1}} df(x_0')$$

which, with Eq. B-33, B-34, and B-37, becomes

$$C(x_0) = A_B f_b\, W_0(0,0) \int_0^{\mu(x_0)} \left[1 - a_0 (\mu'(x_0))^2 \right]^n d\mu'(x_0) . \tag{B-115}$$

Now, let

$$q(\mu) \equiv a_0 \mu^2(x_0)$$

such that

$$d\mu(x_0) = \frac{1}{2} (a_0 q(\mu))^{-\frac{1}{2}} dq(\mu) ,$$

and

$$q(0) = 0.$$

Substituting, we obtain

$$C(x_0) = A_B f_b W_0(0,0) \left\{ \frac{1}{2} a_0^{-\frac{1}{2}} \int_0^{a_0 \mu^2(x_0)} q(\mu)^{-\frac{1}{2}} (1 - q(\mu))^n dq(\mu) \right\} \tag{B-116}$$

or, using Eq. C-17, we have

$$C(x_0) = A_B f_b W_0(0,0)\, \mu(x_0) \left\{ \frac{1}{2} \left(a_0 \mu^2(x_0)\right)^{-\frac{1}{2}} \int_0^{a_0 \mu^2(x_0)} q(\mu)^{-\frac{1}{2}} (1 - q(\mu))^n dq(\mu) \right\}$$

or

$$C(x_0) = A_B f_b W_0(0,0)\, \mu(x_0)\ {}_2F_1\left(-n, \frac{1}{2}; \frac{3}{2}; a_0 \mu^2(x_0)\right). \tag{B-117}$$

Using Eq. B-117 in Eq. B-114 and rearranging terms, we obtain

$$x_0 = \mu(x_0) f_b \left\{ 1 + \eta\, {}_2F_1\left(-n, \frac{1}{2}; \frac{3}{2}; a_0 \mu^2(x_0)\right) \right\} \tag{B-118}$$

which is the desired functional relationship between x_0 and $\mu(x_0)$. The numerical evaluation of this function is discussed in Appendix D.

Summary

The fundamental assumption of the Pidduck-Kent gradient model is that all elements of gas in a Lagrange gun expand along the same adiabatic (K) such that

$$P_0(x_0,0) \left[\frac{1}{\rho_0(x_0,0)} - \eta \right]^\gamma = K \tag{B-2}$$

which is rewritten by defining $W_0(x_0,0)$ such that

$$P_0(x_0,0) = K[W(x_0,0)]^{\gamma} . \qquad \text{(B-13}$$

A new variable is introduced

$$z(x_0,t) = x(x_0,t) - \eta \int_0^{x_0} \rho_0(x_0',0)\, dx_0' = f(x_0)\,\Phi(t) \qquad \text{(B-7), (B-17}$$

and a solution of the momentum equation is attempted by separation of variables. An expression for the total mass of propelling gas in the system is obtained by integrating the gas density distribution over the length of the chamber,

$$C = A_B f_b W_0(0,0)\ {}_2F_1\left(-n, \frac{1}{2}; \frac{3}{2}; a_0\right) \qquad \text{(B-39), (B-42}$$

where $f_b \equiv f(y_0)$, n is the polytropic index,

$$\frac{1}{\gamma - 1} ,$$

and a_0 is a parameter which must be determined in order to satisfy this relation. After some manipulation, a more basic relation is found,

$$2n a_0 (1 - a_0)^{-(n+1)}\ {}_2F_1\left(-n, \frac{1}{2}; \frac{3}{2}; a_0\right) = \varepsilon \qquad \text{(B-47}$$

where ε is the gas propelling charge to projectile mass ratio.

It is shown than each element of gas evolves in time according to the relation

$$\left[W(x_0,t)\right]^{-1} = \left[W_0(x_0,0)\right]^{-1} \Phi(t) . \qquad \text{(B-21}$$

and, more importantly, that the spatial average of $W(x_0, t)$, which is a function of the gas density, evolves in time in an identical manner

$$[\overline{W(t)}]^{-1} = [\overline{W(0)}]^{-1}\, \Phi(t)\ . \tag{B-61}$$

The ratio of the breech pressure to projectile base pressure is found to be

$$\frac{P(0,t)}{P(y_0,t)} = [1 - a_0]^{-(n+1)} \tag{B-50}$$

and the ratio of the space mean pressure to projectile base pressure is

$$\frac{\overline{P(t)}}{P(y_0,t)} = 1 + \frac{\varepsilon}{\delta} \tag{B-80}$$

where

$$\frac{1}{\delta} \equiv \frac{1}{3}\ \frac{{}_2F_1\left(-n, \frac{3}{2}; \frac{5}{2}; a_0\right)}{{}_2F_1\left(-n, \frac{1}{2}; \frac{3}{2}; a_0\right)}\ . \tag{B-81}$$

We note that for $\varepsilon = 0$, $\delta = 3$ such that in the limit of zero C/M, the Pidduck-Kent special solution reduces to that of Lagrange. We also note that in the limit of infinite C/M, $\delta = 2\,n + 3$. A relationship among the parameters, a_0, ε, n, and δ, which is useful for calculating δ, is presented,

$$\frac{1}{\delta} = \frac{1}{2n+3}\left[\frac{1}{a_0} - \frac{2(n+1)}{\varepsilon}\right]. \tag{B-84}$$

An expression for the space mean gas temperature is also presented

$$\overline{T(t)} = T(0,t)\ \frac{{}_2F_1\left(-n-1, \frac{1}{2}; \frac{3}{2}; a_0\right)}{{}_2F_1\left(-n, \frac{1}{2}; \frac{3}{2}; a_0\right)}$$

$$= T(0,t)\left[\frac{2(n+1)\,a_0}{\varepsilon}\right]\left(1 + \frac{\varepsilon}{\delta}\right)\ . \tag{B-72), (B-97}$$

The gas kinetic energy is found to be

$$KE_{GAS} = \left(\frac{\varepsilon}{\delta}\right) KE_{PROJ} \tag{B-110}$$

such that

$$KE_{TOTAL} = \left(1 + \frac{\varepsilon}{\delta}\right) KE_{PROJ}\,, \tag{B-111}$$

which are easily seen to reduce to the Lagrange result in the limit of zero C/M.

The projectile velocity as a function of expansion ratio is determined to be

$$u^2(y_o, t) = \frac{2\,E_o\,\varepsilon}{1 + \frac{\varepsilon}{\delta}} \left[1 - \left(\frac{1}{\varepsilon_f}\right)^{\gamma-1}\right] \tag{B-99}$$

which is the solution of the preburned propellant, nonideal gas gun problem as can be obtained directly from energy considerations. Finally, it is shown that the maximum projectile velocity obtained from the Pidduck-Kent solution of the infinite travel, infinite C/M Lagrange problem is given by

$$u_{max}^{P\text{-}K} = \left[\frac{3\gamma - 1}{2\gamma}\right]^{\frac{1}{2}} \left[\frac{2}{\gamma - 1} \left(\gamma R \overline{T(o)}\right)^{\frac{1}{2}}\right] = \left[\frac{3\gamma - 1}{2\gamma}\right]^{\frac{1}{2}} u_{escape} \tag{B-103, B-104}$$

where u_{escape} is the maximum velocity obtained in the exact solution of this problem.

APPENDIX C:

DISCUSSION OF INTEGRALS ARISING IN PIDDUCK-KENT SPECIAL SOLUTION OF THE LAGRANGE PROBLEM

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Several integrals arise in the development of the Pidduck-Kent special solution of the Lagrange ballistic problem which can be related to the Gauss' hypergeometric functions and incomplete "beta" functions (Arfken 1970; Abramovitz and Stegun). These relationships prove useful in the derivation of the Pidduck-Kent solution and in determining its limiting behavior. The integrals of interest are presented in Appendix B in Eq. B-38, B-68, and B-107,

$$I_1 \equiv \int_o^1 \left(1 - a_0 \mu^2\right)^n d\mu \,, \tag{C-1}$$

$$I_2 \equiv \int_o^1 \left(1 - a_0 \mu^2\right)^{n+1} d\mu \,, \tag{C-2}$$

and

$$I_3 \equiv \int_o^1 \left(1 - a_0 \mu^2\right)^n \mu^2 d\mu \,. \tag{C-3}$$

In Eq. C-1, we define a new variable q such that

$$q \equiv a_0 \mu^2 \tag{C-4}$$

$$\mu = (q/a_0)^{\frac{1}{2}} \tag{C-5}$$

$$d\mu = \frac{1}{2\sqrt{a_0}} q^{-\frac{1}{2}} dq \tag{C-6}$$

$$q(o) = o \tag{C-7}$$

$$q(1) = a_0 \tag{C-8}$$

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* The integral in Eq. B-112 defining $C(x_o)$ is closely related to I_1 and will not be considered separately.

giving us

$$I_1 = \frac{1}{2} a_0^{-\frac{1}{2}} \int_0^{a_0} [1 - q]^n q^{-\frac{1}{2}} dq . \qquad \text{(C-9)}$$

Similarly, Eq. C-2 and C-3 become

$$I_2 = \frac{1}{2} a_0^{-\frac{1}{2}} \int_0^{a_0} [1 - q]^{n+1} q^{-\frac{1}{2}} dq \qquad \text{(C-10)}$$

and

$$I_3 = \frac{1}{2} a_0^{-\frac{3}{2}} \int_0^{a_0} [1 - q]^n q^{\frac{1}{2}} dq . \qquad \text{(C-11)}$$

Gauss' hypergeometric function is defined by (Arfken 1970; Abramovitz and Stegun)

$${}_2F_1(a,b;c;z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a} dt , \qquad \text{(C-12)}$$

where $\Gamma(c)$ is the gamma function. For $c - b = 1$,

$$\Gamma(c - b) = \Gamma(1) = 1 \qquad \text{(C-13)}$$

and,

$$\frac{\Gamma(c)}{\Gamma(b)} = \frac{\Gamma(b+1)}{\Gamma(b)} = \frac{b\Gamma(b)}{\Gamma(b)} = b \qquad \text{(C-14)}$$

such that

$${}_2F_1(a,b;b+1;z) = b \int_0^1 t^{b-1}(1-tz)^{-a} dt . \qquad \text{(C-15)}$$

We now define a new variable, $q = tz$, such that

$${}_2F_1(a,b;b+1;z) = \frac{b}{z^b} \int_0^z q^{b-1}(1-q)^{-a} dq \qquad \text{(C-16)}$$

and, for $a \equiv -n$ and $z \equiv a_0$, we obtain

$$ {}_2F_1(-n, b; b+1; a_0) = \frac{b}{a_0^b} \int_0^{a_0} q^{b-1} (1-q)^n \, dq \qquad \text{(C-17)} $$

which is now in a form which can be used to rewrite I_1, I_2, and I_3 in terms of Gauss hypergeometric functions. Comparing Eq. C-9 and C-17, we find that

$$ I_1 = {}_2F_1\left(-n, \frac{1}{2}; \frac{3}{2}; a_0\right) = \frac{1}{2} a_0^{-\frac{1}{2}} \int_0^{a_0} [1-q]^n q^{-\frac{1}{2}} \, dq \qquad \text{(C-18)} $$

and similarly for Eq. C-10 and C-11,

$$ I_2 = {}_2F_1\left(-n-1, \frac{1}{2}; \frac{3}{2}; a_0\right) = \frac{1}{2} a_0^{-\frac{1}{2}} \int_0^{a_0} [1-q]^{n+1} q^{-\frac{1}{2}} \, dq \qquad \text{(C-19)} $$

and

$$ I_3 = \frac{1}{3} \; {}_2F_1\left(-n, \frac{3}{2}; \frac{5}{2}; a_0\right) = \frac{1}{2} a_0^{-\frac{3}{2}} \int_0^{a_0} [1-q]^n q^{\frac{1}{2}} \, dq \, . \qquad \text{(C-20)} $$

The incomplete "beta" functions are defined in terms of Gauss' hypergeometric functions by (Arfken 1970; Abramovitz and Stegun)

$$ \frac{b}{a_0^b} B_{a_0}(b, n+1) = {}_2F_1(-n, b; b+1; a_0) \qquad \text{(C-21)} $$

such that

$$ I_1 = \frac{1}{2} a_0^{-\frac{1}{2}} B_{a_0}\left(\frac{1}{2}, n+1\right) , \qquad \text{(C-22)} $$

$$I_2 = \frac{1}{2} a_0^{-\frac{1}{2}} B_{a_0}\left(\frac{1}{2}, n+2\right), \tag{C-23}$$

and

$$I_3 = \frac{1}{2} a_0^{-\frac{3}{2}} B_{a_0}\left(\frac{3}{2}, n+1\right). \tag{C-24}$$

We now integrate Eq. C-2 by parts to obtain

$$I_2 = \mu\left[1 - a_0\mu^2\right]^{n+1}\Big|_0^1 + 2(n+1)\,a_0 \int_0^1 \left[1 - a_0\mu^2\right]^n \mu^2\, d\mu\,, \tag{C-25}$$

or

$$I_2 = (1 - a_0)^{n+1} + 2(n+1)\,a_0 I_3\,. \tag{C-26}$$

Alternately, we could proceed by rewriting the integrand in Eq. C-2 to obtain,

$$I_2 = \int_0^1 \left[1 - a_0\mu^2\right]^n d\mu - a_0 \int_0^1 \left(1 - a_0\mu^2\right)^n \mu^2\, d\mu\,, \tag{C-27}$$

or

$$I_2 = I_1 - a_0 I_3\,. \tag{C-28}$$

Equating Eq. C-26 and C-28. we have

$$I_1 - a_0 I_3 = (1 - a_0)^{n+1} + 2(n+1)\,a_0 I_3\,, \tag{C-29}$$

which can be rewritten as

$$I_3 \equiv \frac{1}{2n+3}\left[\frac{I_1}{a_0} - \frac{(1 - a_0)^{n+1}}{a_0}\right]. \tag{C-30}$$

From Eq. B-47, we have

$$(1 - a_0)^{n+1} = \frac{2(n+1)\,a_0 I_1}{\varepsilon} \tag{C-31}$$

and substituting in Eq. C-30, we obtain

$$\frac{I_3}{I_1} = \frac{1}{2n+3}\left[\frac{1}{a_0} - \frac{2(n+1)}{\varepsilon}\right] \tag{C-32}$$

or

$$\frac{1}{3}\,\frac{{}_2F_1\left(-n,\frac{3}{2};\frac{5}{2};a_0\right)}{{}_2F_1\left(-n,\frac{1}{2};\frac{3}{2};a_0\right)} = \frac{1}{2n+3}\left[\frac{1}{a_0} - \frac{2(n+1)}{\varepsilon}\right] \tag{C-33}$$

which is a relationship between two contiguous hypergeometric functions (Arfken 1970; Abramovitz and Stegun). Using Eq. B-81 to identify the left-hand side of Eq. C-33 as $\frac{1}{\delta}$, we have

$$\frac{1}{\delta} = \frac{1}{2n+3}\left[\frac{1}{a_0} - \frac{2(n+1)}{\varepsilon}\right]. \tag{C-34}$$

Eq. C-28, also an equation for contiguous functions, can be rewritten as

$$\frac{{}_2F_1\left(-n-1,\frac{1}{2};\frac{3}{2};a_0\right)}{{}_2F_1\left(-n,\frac{1}{2};\frac{3}{2};a_0\right)} = 1 - \frac{a_o}{3}\,\frac{{}_2F_1\left(-n,\frac{3}{2};\frac{5}{2};a_0\right)}{{}_2F_1\left(-n,\frac{1}{2};\frac{3}{2};a_0\right)} = 1 - \frac{a_0}{\delta}\,. \tag{C-35}$$

Following the arguments of Vinti and Kravitz (1949), we first note that, from Eq. B-50,

$$\frac{P(y_o,t)}{P(o,t)} = \left[1 - a_0\right]^{(n+1)} \tag{C-36}$$

it is clear that $0 \le a_0 \le 1$. Since $n \equiv \frac{1}{\gamma - 1}$ and $\gamma > 1$, then n is a real, positive number. Thus, if the pressure ratio given in Eq. B-50 is to be a real number and $P(o,t) \ge P(y_o,t)$, it must follow that $0 \le a_0 \le 1$. We also note that the integrals of interest, Eq.s C-1, C-2, and C-3 are real and finite for $n \ge o$ and $0 \le a_0 \le 1$. Then, from Eq. B-47

$$\varepsilon = 2(n+1)\,a_0(1-a_0)^{-(n+1)}\,{}_2F_1\left(-n,\frac{1}{2};\frac{3}{2};a_0\right) \tag{C-37}$$

we see that $a_0 = 0$ corresponds to the limit of zero C/M while $a_0 = 1$ corresponds to the limit of infinite C/M. Using Eq. C-15 to evaluate the hypergeometric function for $a_0 = 0$, we have

$${}_2F_1(a,b;b+1;0) = b\int_0^1 t^{b-1}\,dt = t^b\Big|_o^1 = 1\ . \tag{C-38}$$

For $a_0 = 1$, the integrals of interest reduce to regular (complete) "beta" functions, giving us

$$I_1(a_0=1) = \frac{1}{2}B\left(\frac{1}{2},n+1\right) = \frac{1}{2}\frac{\Gamma\left(\frac{1}{2}\right)\Gamma(n+1)}{\Gamma\left(n+\frac{3}{2}\right)}$$

$$= \frac{n}{2\left(n+\frac{1}{2}\right)}\frac{\Gamma\left(\frac{1}{2}\right)\Gamma(n)}{\Gamma\left(n+\frac{1}{2}\right)}, \tag{C-39}$$

$$I_2(a_0=1) = \frac{1}{2}B\left(\frac{1}{2},n+2\right) = \frac{1}{2}\frac{\Gamma\left(\frac{1}{2}\right)\Gamma(n+2)}{\Gamma\left(n+\frac{5}{2}\right)}$$

$$= \frac{n(n+1)}{2\left(n+\frac{3}{2}\right)\left(n+\frac{1}{2}\right)}\frac{\Gamma\left(\frac{1}{2}\right)\Gamma(n)}{\Gamma\left(n+\frac{1}{2}\right)}, \tag{C-40}$$

and

$$I_3(a_0 = 1) = \frac{1}{2} B\left(\frac{3}{2}, n+1\right) = \frac{1}{2} \frac{\Gamma\left(\frac{3}{2}\right)\Gamma(n+1)}{\Gamma\left(n+\frac{5}{2}\right)}$$

$$= \frac{\left(\frac{1}{2}\right)n}{2\left(n+\frac{3}{2}\right)\left(n+\frac{1}{2}\right)} \frac{\Gamma\left(\frac{1}{2}\right)\Gamma(n)}{\Gamma\left(n+\frac{1}{2}\right)} . \qquad \text{(C-41)}$$

Recalling that

$$\frac{1}{\delta} \equiv \frac{1}{3} \frac{{}_2F_1\left(-n, \frac{3}{2}; \frac{5}{2}; a_0\right)}{{}_2F_1\left(-n, \frac{1}{2}; \frac{3}{2}; a_0\right)} = \frac{I_3}{I_1} , \qquad \text{(C-42)}$$

we obtain for $a_0 = 0$

$$\frac{1}{\delta} = \frac{1}{3} \qquad \text{(C-43)}$$

and for $a_0 = 1$,

$$\frac{1}{\delta} = \frac{B\left(\frac{3}{2}, n+1\right)}{B\left(\frac{1}{2}, n+1\right)} = \frac{1}{2n+3} . \qquad \text{(C-44)}$$

This latter relation can be obtained directly from Eq. C-34. However, the zero C/M limit is shown to be a direct result of the solution, in contrast to the argument used by Vinti and Kravitz (1949) involving expansion of ε in Eq. C-31 in terms of a_0 and substitution of the result into Eq. C–34.

The ratio I_2/I_1 arises in Eq. B–72 relating the space mean gas temperature to the temperature at the breech. For $a_0 = 1$, we obtain

$$\frac{I_2(a_0 = 1)}{I_1(a_0 = 1)} = \frac{(n+1)}{\left(n+\frac{3}{2}\right)} = \frac{2(n+1)}{2n+3} = \frac{2\gamma}{\gamma - 1} , \qquad \text{(C-45)}$$

which is the factor arising in Eq. B-101 and B-102 relating the maximum velocity from the Pidduck-Kent solution for the infinite travel, infinite C/M Lagrange gun problem and the escape velocity obtained from the exact solution of this problem.

APPENDIX D:

NUMERICAL EVALUATION OF THE PIDDUCK-KENT PARAMETERS

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The evaluation of the Pidduck-Kent parameters, a_0 and δ, defined by Eq. B-47 (or C-37) and Eq. B-82,

$$\varepsilon = 2(n+1)\,a_0(1-a_0)^{-(n+1)}\;{}_2F_1\left(-n,\frac{1}{2};\frac{3}{2};a_0\right) \tag{D-1}$$

and

$$\frac{1}{\delta} = \frac{1}{2n+3}\left[\frac{1}{a_0} - \frac{2(n+1)}{\varepsilon}\right] \tag{D-2}$$

where

$${}_2F_1\left(-n,\frac{1}{2};\frac{3}{2};a_0\right) = \int_o^1 \left(1 - a_0\mu^2\right)^n d\mu \tag{D-3}$$

has been considered previously by Vinti and Kravitz (1949) and by Grollman and Baer (1970). Vinti and Kravitz developed an algebraic expression for δ as a function of n,

$$\frac{1}{\delta} = \frac{1}{2n+3}\left[1 + \alpha(\varepsilon)\,n\left(\frac{1 + C_1(\varepsilon,n)\,\beta(\varepsilon)\,n}{1 + C_1(\varepsilon,n)\,n}\right)\right] \tag{D-4}$$

where the coefficients $\alpha(\varepsilon)$, $\beta(\varepsilon)$, and $C_1(\varepsilon,n)$ are themselves functions of ε and n. These coefficients are tabulated (Vinti and Kravitz 1949) for selected values of ε and n such that the value of the coefficients, and thus, δ can be determined for arbitrary ε ($0 \le \varepsilon \le 10$) and n ($0.5 \le n \le 5$ [i.e., $1.2 \le \gamma \le 3$]) by interpolation. While the computed values of δ obtained using the tables of Vinti and Kravitz are quite accurate, this technique is cumbersome and not well-suited for application in computer simulations. In contrast, Grollman and Baer developed extensive tables for δ as a function of ε and n, for $0 \le \varepsilon \le 20$, and $1.2 \le \gamma \le 1.3$, by numerical integration to obtain ε as a function of a_0 for given n (see Eq. D-1 and D-3) and use of Eq. D-2 to obtain δ. A least squares fit of the tabulated values of δ was then generated using a fifth order polynomial function of ε and γ,

$$\delta = 3.0 + \sum_{i=1}^{5}\left[\alpha_i + \frac{\beta_i}{\gamma}\right]\varepsilon^i \tag{D-5}$$

This expression is easily applied in interior ballistic simulations, and has served as the primary means for implementation of the Pidduck-Kent gradient model. However, in using Eq. D-5 to compute δ, errors on the order of a few percent are introduced. Additionally, the range of applicability of this equation is limited, particularly in view of the properties of some novel propellants being considered in conjunction with the electrothermal-chemical gun.

Our objective, then, is to develop a general technique for calculation of the Pidduck-Kent parameters, a_0 and δ, which is valid for arbitrary ε and δ, and which is readily applicable in computer simulations. As a by-product, we also develop techniques to evaluate the other functions encountered in the Pidduck-Kent solution of the Lagrange problem.

The approach selected to develop a generally applicable method for determination of the Pidduck-Kent parameters is direct solution of Eq. D-1 for a_0. The hypergeometric function may be expressed as an infinite series of the form,

$$9\ {}_2F_1(\phi,\psi;\theta;x) = \frac{\Gamma(\theta)}{\Gamma(\phi)\Gamma(\psi)} \sum_{k=0}^{\infty} \frac{\Gamma(k+\phi)\Gamma(k+\psi)}{\Gamma(k+\theta)} \frac{x^k}{k!} \qquad \text{(D-6)}$$

where the series converges for $\theta - \psi - \phi > 0$ and $|x| \leq 1$. For $\theta = \psi + 1$, the Gauss hypergeometric function reduces to the incomplete "beta" function for $0 \leq x \leq 1$ and the regular (complete) "beta" function for $x = 1$. For the functions of interest in the Pidduck-Kent solution, $\theta = \psi + 1$ and $\phi = -n$ or $-(n+1)$ where θ, ψ and m are positive numbers such that $\theta - \psi - \phi > 0$. We also note that $x = a_0$ with $0 \leq a_0 \leq 1$, such that the series in Eq. D-6 converges for the values of parameters considered here.

We are generally interested in Gauss hypergeometric functions of the form

$${}_2F_1(1-b,a;a+1;x) = \frac{\Gamma(a+1)}{\Gamma(1-b)\Gamma(a)} \sum_{k=0}^{\infty} \frac{\Gamma(k+1-b)\Gamma(k+a)}{\Gamma(k+1+a)} \frac{x^k}{k!} . \qquad \text{(D-7)}$$

We will now simplify this expression using the properties of the "gamma" function. First, we note that

$$\frac{\Gamma(a+1)}{\Gamma(a)} = \frac{a\Gamma(a)}{\Gamma(a)} = 2 \tag{D-8}$$

and

$$\frac{\Gamma(k+a)}{\Gamma(k+1+a)} = \frac{\Gamma(k+a)}{(k+a)\Gamma(k+a)} = \frac{1}{k+a} . \tag{D-9}$$

We now rewrite Eq. D-7, to obtain

$${}_2F_1(1-b,a;a+1;x) = 1 + a\sum_{k=1}^{\infty} \frac{\Gamma(k+1-b)}{\Gamma(1-b)} \frac{x^k}{(k+a)\,k!} . \tag{D-10}$$

The ratio of "gamma" functions can be expanded to give us

$$\frac{\Gamma(k+1-b)}{\Gamma(1-b)} = \frac{1}{\Gamma(1-b)}\{(k-b)(k-b-1).....(1-b)\Gamma(1-b)\}$$

or

$$\frac{\Gamma(k+1-b)}{\Gamma(1-b)} = \prod_{j=1}^{k}(k+1-b-j) \tag{D-11}$$

such that

$${}_2F_1(1-b,a;a+1;x) = 1 + \sum_{k=1}^{\infty}\prod_{j=1}^{k}(k+1-b-j)\left(\frac{a}{k+a}\right)\frac{x^k}{k!} . \tag{D-12}$$

Eq. D-1 can now be solved iteratively using the Newton-Raphson technique. We define

$$G(a_0,n;\varepsilon) = 2(n+1)\,a_0(1-a_0)^{-(n+1)}\ {}_2F_1\left(-n,\frac{1}{2};\frac{3}{2};a_0\right) - \varepsilon = 0 \tag{D-13}$$

such that

$$\frac{\partial G(a_0,n,\varepsilon)}{\partial a_0} = \left[G(a_0,n;\varepsilon) + \varepsilon \right] \left\{ \frac{1}{a_0} + \frac{n+1}{1-a_0} \right\}$$

$$+ 2(n+1)a_0(1-a_0)^{-(n+1)} \left\{ \frac{\partial_2 F_1\left(-n, \frac{1}{2}; \frac{3}{2}; a_0\right)}{\partial a_0} \right\} \qquad \text{(D-14)}$$

where

$${}_2F_1\left(-n, \frac{1}{2}; \frac{3}{2}; a_0\right) = 1 + \sum_{k=1}^{\infty} \prod_{j=1}^{k} (k-n-j) \frac{1}{2k+1} \frac{a_0^k}{k!} \qquad \text{(D-15)}$$

and

$$\frac{\partial_2 F_1\left(-n, \frac{1}{2}; \frac{3}{2}; a_0\right)}{\partial a_0} = \sum_{k=1}^{\infty} \prod_{j=1}^{k} (k-n-j) \frac{k}{2k+1} \frac{a_0^{k-1}}{k!} . \qquad \text{(D-16)}$$

An initial estimate for a_0 is obtained using the Grollman and Baer (1970) formula (Eq. D-5) to calculate $\delta^{(1)}$ and Eq. D-2 to calculate $a_0^{(1)}$. Successive values of $a_0^{(i)}$ are then calculated iteratively using the Newton-Raphson equation

$$a_0^{(i+1)} = a_0^{(i)} - G\left(a_0^{(i)}, n, \varepsilon\right) \left[\left. \frac{\partial G(a_0,n,\varepsilon)}{\partial a_0} \right|_{a_0^{(i)}} \right]^{-1} \qquad \text{(D-17)}$$

until the desired accuracy is achieved.

In order to improve convergence of the infinite series in Eq. D-7 for $0.65 \le x \le 1.0$, we consider the integral representation of the hypergeometric function

$${}_2F_1(1-b, a; a+1; x) = \frac{a}{x^a} \int_0^x t^{a-1} (1-t)^{b-1} dt . \qquad \text{(D-18)}$$

Changing variables, we define

$$q \equiv (1 - t)$$

such that

$$dq = - dt$$

$$q(0) = 1$$

and

$$q(x) = 1 - x$$

giving us

$${}_2F_1(1 - b, a; a + 1; x) = \frac{a}{x^a} \int_{1-x}^{1} (1 - q)^{a-1} q^{b-1} dq$$

or

$${}_2F_1(1 - b, a; a + 1; x) = \frac{a}{x^a} \left\{ \int_0^1 (1 - q)^{a-1} q^{b-1} dq - \int_0^{1-x} (1 - q)^{a-1} q^{b-1} dq \right\}.$$

Using Eq. D-18 and rewriting, we have

$$\frac{x^a}{a} {}_2F_1(1 - b, a; a + 1; x) = B(b, a) - \frac{(1 - x)^b}{b} {}_2F_1(1 - a, b; b + 1; 1 - x) \qquad \text{(D-19)}$$

where $B(b,a)$ is the regular (or complete) "beta" function. Recalling that the incomplete "beta" function and Gauss hypergeometric function are related by

$$B_x(a, b) = \frac{x^a}{a} {}_2F_1(1 - b, a; a + 1; x) \qquad \text{(D-20)}$$

we obtain from Eq. D-19,

$$B_x(a, b) = B(b, a) - B_{1-x}(b, a) \qquad \text{(D-21)}$$

which (Abramovitz and Stegun) we might have used as the starting point to obtain the desired relation, Eq. D-19. From the symmetry of the "beta" function, i.e.,

$$B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} = B(b,a) , \tag{D-22}$$

we see that Eq. D-21 can be rewritten as

$$B_x(a,b) = B(a,b) - B_{1-x}(b,a) \tag{D-23}$$

and that $B(b,a)$ could be similarly replaced in Eq. D-19.

Returning now to Eq. D-19 for the arguments of interest, we have

$$2\, a_0^{\frac{1}{2}}\, {}_2F_1\left(-n, \frac{1}{2}; \frac{3}{2}; a_0\right) = B\left(\frac{1}{2}, n+1\right)$$

$$- \frac{(1\ a_0)^{n+1}}{n+1}\ {}_2F_1\left(\frac{1}{2}; n+1; n+2; 1-a_0\right) \tag{D-24}$$

where

$$B\left(\frac{1}{2}, n+1\right) = \frac{\Gamma\left(\frac{1}{2}\right)\Gamma(n)}{\Gamma\left(n+\frac{3}{2}\right)} , \tag{D-25}$$

$$\Gamma\left(\frac{1}{2}\right) = \pi^{\frac{1}{2}} , \tag{D-26}$$

and

$$ {}_2F_1\left(\frac{1}{2};n+1;n+2;1-a_0\right) = 1 + \sum_{k=1}^{\infty}\prod_{j=1}^{k}\left(k+\frac{1}{2}-j\right) \times \frac{n+1}{k+n+1}\,\frac{(1-a_0)^k}{k!}. \qquad \text{(D-27)} $$

We also require the derivative with respect to a_0,

$$ \frac{\partial\, {}_2F_1\left(\frac{1}{2};n+1;n+2;1-a_0\right)}{\partial a_0} = -\sum_{k=1}^{\infty}\prod_{j=1}^{k}\left(k+\frac{1}{2}-j\right) \times \frac{n+1}{k+n+1}\,\frac{k(1-a_0)^{k-1}}{k!}. \qquad \text{(D-28)} $$

Therefore, for $a_0 > 0.65$, Eq. D-13 and D-14 become

$$ G(a,n,\varepsilon) = 2(n+1)\,a_0(1-a_0)^{-(n+1)}\left\{\frac{1}{2}a_0^{-\frac{1}{2}}\left[B\left(\frac{1}{2},n+1\right) - \frac{(1-a_0)^{n+1}}{n+1}\,{}_2F_1\left(\frac{1}{2};n+1;n+2;1-a_0\right)\right]\right\} - \varepsilon \qquad \text{(D-29)} $$

and

$$ \frac{\partial G(a_0,n,\varepsilon)}{\partial a_0} = \left[G(a_0,n,\varepsilon)-\varepsilon\right]\left[\frac{1}{a_0}+\frac{n+1}{1-a_0}-\frac{1}{2a_0}\right] + 2(n+1)\,a_0(1-a_0)^{-(n+1)}\left\{\frac{1}{2}a_0^{-\frac{1}{2}}(1-a_0)^n\left[{}_2F_1\left(\frac{1}{2};n+1;n+2;1-a_0\right) - \frac{1-a_0}{n+1}\,\frac{\partial\, {}_2F_1\left(\frac{1}{2};n+1;n+2;1-a_0\right)}{\partial a_0}\right]\right\} \qquad \text{(D-30)} $$

In practice, a_0 is generally less than 0.5 for the values of ε and γ of interest. However, we do utilize this alternate formulation in comparison of a_0 and δ calculated using the other techniques (Vinti and Kravitz 1949; Grollman and Baer 1970).

The computational technique described above has been implemented in a computer program, which is presented in Appendix F. This program has been used to calculate a_0 and δ for the values of ε and n from the tables of Vinti and Kravitz (1949). The results of these calculations are presented in Tables D-1–D-4, along with corresponding values of a_0 and δ obtained using the techniques developed by Vinti and Kravitz (1949) and Grollman and Baer (1970). Comparing the values of a_0 and δ calculated using the techniques presented here with those of Vinti and Kravitz, we see that they generally agree within 1 part in 10^5. We note that Vinti and Kravitz (1949) have estimated that values of a_0 and δ calculated from their tables have an accuracy of 1 part in 6,700 and 1 part in 5,000, respectively. The convergence criteria for the Newton-Raphson iteration used in our calculations provides an accuracy of at least 1 part in 10^5 in the determination of a_0. Therefore, the technique presented here is at least as accurate as that of Vinti and Kravitz over the range of ε and γ considered. As noted, this technique has been implemented in a computer program, therefore, our initial objective has been accomplished. We also note that the accuracy of the Grollman and Baer (1970) formula is somewhat less than that of the other two approaches, varying fromm about 1 part in 300 for charge-to-mass ratios around 3, to about 1 part in 1000 at higher and lower charge-to-mass ratios.

Grollman and Baer (1970) have published tables of the Pidduck-Kent parameters (δ) for $0 \leq \varepsilon \leq 26$ in increments of 0.01 and for $1.2 \leq \gamma \leq 1.3$ in increments of 0.01. These results are based on a numerical integration to obtain ε as a function of a_0 for given n and use of Eq. D-2 to calculate δ. The values of δ at the tabulated points were obtained by interpolation. Selected values of δ from these tables are presented in Tables D-5–D-7, along with corresponding values of δ calculated using the least squares fit of Grollman and Baer (1970) and the technique developed in this work in order to assess the accuracy of the current computational method for $10 \leq \varepsilon \leq 20$. As can be seen from these tables, our values agree quite well with the tabulated values of Grollman and Baer. The accuracy of the Grollman and Baer least squares fit is comparable to that noted earlier at lower charge-to-mass ratios. Finally, we note that the accuracy of the tabulated values of δ from Grollman and Baer is

5 parts in 10^5 (based on the convergence criteria for their numerical integrator) which is consistent with comparisons to values of δ obtained using the technique developed in this work.

As an additional note, we point out other applications of the techniques developed in this Appendix. In order to calculate x_0 as a function of μ for $0 \leq \mu \leq 1$ from Eq. B-118, or to evaluate the other hypergeometric functions arising in the Pidduck-Kent solution (i.e., Eq. C-19 and C-20) we can utilize Eq. D-12 and D-19.

The computer program used to calculate a_0 and δ with the numerical method described above has been incorporated into the lumped parameter, interior ballistic code, IBHVG2. IBHVG2 is a standard, well-documented, IB model for solid propellant guns containing the Grollman and Baer least squares fit for ε. The IB code was modified to permit the use of either the least squares fit or the numerical technique described in this report to determine a_0 and δ.

Since the accuracy of the least squares fit decreases with increasing ε, a sample problem of interest with charge-to-mass ratio of 3.003 was examined. A 120-mm gun with 7 perf, JA2 propellant was chosen as a baseline. The values of a_0, δ, maximum breech pressure and muzzle velocity for each case is shown in Table D-1. Three cases are presented: (1) least squares fit with an optimized web; (2) numeric method using the web determined in (1); and (3) numeric method with an optimized web.

A comparison of the three cases in Table D-8 shows that the numeric method gives approximately a 0.1% difference in maximum breech pressure and a 0.03% difference in muzzle velocity compared to the least squares fit for calculating δ for a given, optimized web. However, if the web is re-optimized, the differences drop to 0.007% in maximum breech pressure and 0.004% in muzzle velocity. Thus, for high charge-to-mass ratios in a regime of interest (i.e., ~ 3 in this application) differences in predicted gun performance using fitted or numerically calculated values for δ and a_0 in the Pidduck-Kent gradient are considered insignificant.

Table D-1. Comparison of Pidduck-Kent Parameters for $\gamma = 1.2$

	Vinti and Kravitz (1949)		Grollman and Baer (1970)		Morrison et al.	
ε	a_0	δ	a_0	δ	a_0	δ
0.2	0.01557	3.06355	0.01556	3.05887	0.01557	3.06352
0.4	0.02927	3.12164	0.02926	3.11479	0.02927	3.12159
0.6	0.04150	3.17532	0.04149	3.16790	0.04150	3.17517
0.8	0.05255	3.22502	0.05252	3.21837	0.05255	3.22497
1.0	0.06260	3.27150	0.06258	3.26633	0.06260	3.27156
2.0	0.10259	3.46897	0.10264	3.47325	0.10259	3.46887
3.0	0.13184	3.62597	0.13201	3.63614	0.13184	3.62604
4.0	0.15481	3.75762	0.15502	3.76739	0.15481	3.75754
5.0	0.17365	3.87054	0.17381	3.87664	0.17366	3.87099
6.0	0.18963	3.97123	0.18962	3.97115	0.18962	3.97099
7.0	0.20343	4.06084	0.20328	4.05609	0.20342	4.06053
8.0	0.21558	4.14194	0.21533	4.13486	0.21557	4.14168
9.0	0.22641	4.21608	0.22616	4.20939	0.22640	4.21593
10.0	0.23616	4.28408	0.23601	4.28047	0.23617	4.28440

Table D-2. Comparison of Pidduck-Kent Parameters for $\gamma = 1.25$

	Vinti and Kravitz (1949)		Grollman and Baer (1970)		Morrison et al.	
ε	a_0	δ	a_0	δ	a_0	δ
0.2	0.01866	3.06099	0.01866	3.05648	0.01866	3.06097
0.4	0.03505	3.11672	0.03504	3.11009	0.03505	3.11666
0.6	0.04966	3.16816	0.04964	3.16101	0.04966	3.16801
0.8	0.06281	3.21577	0.06279	3.20936	0.06281	3.21572
1.0	0.07477	3.26027	0.07474	3.25529	0.07477	3.26033
2.0	0.12211	3.44893	0.12217	3.45312	0.12211	3.44884
3.0	0.15649	3.59850	0.15670	3.60841	0.15649	3.59856
4.0	0.18335	3.72355	0.18360	3.73312	0.18334	3.72347
5.0	0.20527	3.83054	0.20546	3.83662	0.20528	3.83093
6.0	0.22378	3.92579	0.22379	3.92591	0.22377	3.92553
7.0	0.23972	4.01030	0.23955	4.00601	0.23971	4.01001
8.0	0.25370	4.08667	0.25342	4.08019	0.25369	4.08644
9.0	0.26612	4.15634	0.26585	4.15033	0.26612	4.15624
10.0	0.27727	4.22015	0.27713	4.21718	0.27729	4.22051

Table D-3. Comparison of Pidduck-Kent Parameters for $\gamma = 1.3333$

	Vinti and Kravitz (1949)		Grollman and Baer (1970)		Morrison et al.	
ε	a_0	δ	a_0	δ	a_0	δ
0.2	0.02329	3.05716	0.02328	3.05288	0.02328	3.05713
0.4	0.04368	3.10933	0.04366	3.10306	0.04368	3.10928
0.6	0.06179	3.15745	0.06176	3.15066	0.06179	3.15732
0.8	0.07806	3.20194	0.07802	3.19584	0.07805	3.20190
1.0	0.09281	3.24349	0.09277	3.23872	0.09280	3.24354
2.0	0.15078	3.41909	0.15083	3.42293	0.15077	3.41902
3.0	0.19244	3.55765	0.19267	3.56682	0.19243	3.55772
4.0	0.22471	3.67305	0.22498	3.68175	0.22469	3.67296
5.0	0.25085	3.77136	0.25104	3.77661	0.25085	3.77177
6.0	0.27279	3.85854	0.27275	3.85808	0.27276	3.85837
7.0	0.29158	3.93577	0.29132	3.93091	0.29155	3.93552
8.0	0.30797	4.00526	0.30758	3.99821	0.30794	4.00510
9.0	0.32248	4.06864	0.32207	4.06176	0.32245	4.06849
10.0	0.33545	4.12640	0.33518	4.12228	0.33544	4.12670

Table D-4. Comparison of Pidduck-Kent Parameters for $\gamma = 1.5$

	Vinti and Kravitz (1949)		Grollman and Baer (1970)		Morrison et al.	
ε	a_0	δ	a_0	δ	a_0	δ
0.2	0.03097	3.05078	0.03096	3.04690	0.03097	3.05075
0.4	0.05794	3.09705	0.05792	3.09132	0.05794	3.09700
0.6	0.08177	3.13964	0.08174	3.13341	0.08177	3.13954
0.8	0.10307	3.17900	0.10303	3.17329	0.10307	3.17895
1.0	0.12230	3.21565	0.12225	3.21108	0.12230	3.21569
2.0	0.19696	3.36984	0.19702	3.37259	0.19696	3.36978
3.0	0.24966	3.49055	0.24991	3.49745	0.24966	3.49060
4.0	0.28988	3.59030	0.29014	3.59607	0.28988	3.59025
5.0	0.32208	3.67490	0.32217	3.67655	0.32209	3.67514
6.0	0.34879	3.74927	0.34853	3.74497	0.34878	3.74912
7.0	0.37147	3.81492	0.37085	3.80567	0.37145	3.81466
8.0	0.39107	3.87361	0.39020	3.86150	0.39106	3.87348
9.0	0.40829	3.92687	0.40732	3.91406	0.40828	3.92682
10.0	0.42358	3.97536	0.42267	3.96401	0.42360	3.97560

Table D-5. Comparisons of Pidduck-Kent Parameter δ;
$\gamma = 1.2$ and $\gamma = 1.22$ for $\varepsilon \geq 10$.

	$\gamma = 1.2$			$\gamma = 1.22$		
	Grollman and Baer (1970)		Morrison et al.	Grollman and Baer (1970)		Morrison et al.
ε	CALC	FIT		CALC	FIT	
10	4.2844	4.2805	4.28440	4.2581	4.2545	4.25808
12	4.4073	4.4115	4.40727	4.3777	4.3824	4.37773
14	4.5153	4.5233	4.51525	4.4828	4.4912	4.48276
16	4.6116	4.6128	4.61164	4.5764	4.5781	4.57642
18	4.6987	4.6886	4.69871	4.6610	4.6516	4.66095
20	4.7781	4.7811	4.77814	4.7380	4.7416	4.73799

Table D-6. Comparisons of Pidduck-Kent Parameter δ;
$\gamma = 1.24$ and $\gamma = 1.26$ for $\varepsilon \geq 10$.

	$\gamma = 1.24$			$\gamma = 1.26$		
	Grollman and Baer (1970)		Morrison et al.	Grollman and Baer (1970)		Morrison et al.
ε	CALC	FIT		CALC	FIT	
10	4.2328	4.2294	4.23279	4.2085	4.2051	4.20846
12	4.3474	4.3541	4.34935	4.3221	4.3268	4.32208
14	4.4516	4.4602	4.45157	4.4216	4.4301	4.42161
16	4.5426	4.5446	4.54263	4.5102	4.5121	4.51020
18	4.6248	4.6158	4.62475	4.5900	4.5812	4.59001
20	4.6995	4.7035	4.69952	4.6626	4.6665	4.66264

Table D-7. Comparisons of Pidduck-Kent Parameter δ; $\gamma = 1.28$ and $\gamma = 1.30$ for $\varepsilon \geq 10$.

	$\gamma = 1.28$			$\gamma = 1.30$		
	Grollman and Baer (1970)		Morrison et al.	Grollman and Baer (1970)		Morrison et al.
ε	CALC	FIT		CALC	FIT	
10	4.1850	4.1816	4.18504	4.1625	4.1588	4.16248
12	4.2958	4.3003	4.29584	4.2706	4.2746	4.27059
14	4.3928	4.4009	4.39282	4.3651	4.3727	4.36512
16	4.4790	4.4806	4.47904	4.4491	4.4502	4.44909
18	4.5567	4.5476	4.55666	4.5246	4.5151	4.52462
20	4.6272	4.6307	4.62724	4.5932	4.5960	4.59324

Table D-8. Comparison of Influence of Calculating δ and a_0 Using Least Squares Fit vs. Numeric Method on Gun Performance

Calculation	δ	a_0	Max Breech Pressure (MPa)	Muzzle Velocity (m/s)
Fit (optimized web)	3.62239456	0.14475450	690.0536	2461.12
Numerical method (same web)	3.61239195	0.14456433	689.2609	2460.30
Numerical method (optimized web)	3.61239195	0.14456433	690.1069	2461.03

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APPENDIX E:

LIMITING VELOCITY FOR PREBURNED PROPELLANT, NONIDEAL GAS (PPNIG) GUN

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The statement of the preburned propellant, nonideal gas (PPNIG) gun problem is identical to that of the Lagrange ballistic problem. It is assumed that the gun is a right circular cylinder (i.e., no chambrage) closed at one end by the breech. (Siegel [1979] relaxes this assumption and treats preburned propellant gun cases with and without chambrage. For our purposes we consider only the latter.) Prior to the start of projectile motion, the propellant is assumed to be instantaneously burned, such that the chamber is initially filled with a quiescent gas of uniform pressure, density, and temperature. The gas is assumed to be inviscid and heat loss to the walls is neglected, as are losses due to resistive forces (i.e., the flow is assumed to be isentropic). The solution of this problem can be accomplished either by use of a gradient model, as in Appendices A and B, or by numerical integration of the equations of motion for the system. If, however, the objective is to determine a limiting velocity for this problem, it is assumed that the projectile travel and charge-to-mass ratio (C/M) are permitted to approach infinity. The solution of the resulting problem is then straightforward.

Characteristic Solution

The characteristic equations for an effectively infinite travel, infinite C/M preburned propellant gun is presented by Siegel (1979) for the ideal gas case. An extension to the nonideal (Nobel-Abel) gas equation of state is presented, following the development of Siegel.

When the projectile begins to move, a rarefaction wave moves back into the quiescent gas at the speed of sound, c_o. If the chamber is sufficiently long, either there will be no wave reflections from the breech or the reflected wave will not reach the projectile base prior to muzzle exit. The equation of state of the gas is

$$P_o\left(\frac{1}{\rho_o} - \eta\right) = RT_o\,, \tag{E-1}$$

the gas internal specific energy is

$$E_o = \frac{RT_o}{\gamma - 1}\,, \tag{E-2}$$

and the gas sound speed is

$$c_o^2 = \frac{\gamma P_o}{\rho_o(1 - \eta\rho_o)} = \frac{\gamma R T_o}{(1 - \eta\rho_o)^2} . \tag{E-3}$$

We note that, consistent with our assumptions regarding the initial state of the system, the thermodynamic properties of the gas are uniform throughout the gas such that space mean and local values of any quantity are equal. Since the gas is assumed to expand isentropically as a result of projectile motion,

$$P(x,t)\left(\frac{1}{\rho(x,t)} - \eta\right)^{\gamma} = K = P_o\left(\frac{1}{\rho_o} - \eta\right)^{\gamma} . \tag{E-4}$$

The Riemann function, ξ, is defined by

$$d\xi \equiv \left(\frac{dP}{c\rho}\right)_s = \left(\frac{c d\rho}{\rho}\right)_s \tag{E-5}$$

where *s* denotes constant entropy. Since we have assumed the expansion process is isentropic, this subscript will be dropped. We note that the definition of the sound speed has been used in Eq. E-5 to relate dP and $d\rho$, i.e.,

$$c^2 \equiv \left(\frac{\partial P}{\partial \rho}\right)_s . \tag{E-6}$$

Using Eq. E-1, E-3, and E-4, it can be shown that

$$c\rho = \gamma P_o(\gamma R T_o)^{-\frac{1}{2}}\left(\frac{P}{P_o}\right)^{\frac{\gamma+1}{2\gamma}} . \tag{E-7}$$

Substituting Eq. E-7 in Eq. E-5 and integrating, we obtain

$$\xi = \frac{(\gamma R T_o)^{\frac{1}{2}}}{\gamma P_o} P_o^{\frac{\gamma+1}{2\gamma}} \int P^{-\left(\frac{\gamma+1}{2\gamma}\right)} dP \tag{E-8}$$

or

$$\xi = \frac{(\gamma R T_o)^{\frac{1}{2}}}{\gamma P_o} \left(\frac{P_o}{P}\right)^{\frac{\gamma+1}{2\gamma}} \left[\frac{2\gamma P}{\gamma - 1}\right] = \left(\frac{2}{\gamma - 1}\right) \frac{\gamma P}{c\rho} . \tag{E-9}$$

From Eq. E-3 we note that

$$\frac{\gamma P}{c\rho} = c(1 - \eta\rho) = (\gamma R T)^{\frac{1}{2}} \tag{E-10}$$

such that

$$\xi = \left(\frac{2}{\gamma - 1}\right) (\gamma R T)^{\frac{1}{2}} . \tag{E-11}$$

The identical result is obtained in the case of an ideal gas, where $c^2 \equiv \gamma RT$.

The characteristic solution for the system is

$$u + \xi = \xi_o \tag{E-12}$$

where $u\,(P_o, \xi_o) \equiv 0$. It is convenient to evaluate ξ and ξ_o using Eq. E-9 which can be rewritten in the form

$$\xi = \left(\frac{2}{\gamma - 1}\right) (\gamma R T_o)^{\frac{1}{2}} \left(\frac{P}{P_o}\right)^{\frac{\gamma-1}{2\gamma}} \tag{E-13}$$

such that Eq. E-12 becomes

$$u = \left(\frac{2}{\gamma - 1}\right)(\gamma R T_o)^{\frac{1}{2}}\left[1 - \left(\frac{P}{P_o}\right)^{\frac{\gamma-1}{2\gamma}}\right] \tag{E-14}$$

or

$$\frac{P}{P_o} = \left[1 - \frac{(\gamma - 1)u}{2(\gamma R T_o)^{\frac{1}{2}}}\right]^{\frac{2\gamma}{\gamma-1}} . \tag{E-15}$$

When the gas pressure at the base of the projectile drops to zero, the gas can do no more work and the projectile has attained the maximum possible velocity or the "escape velocity,"

$$u_{escape} = \left(\frac{2}{\gamma - 1}\right)(\gamma R T_o)^{\frac{1}{2}} . \tag{E-16}$$

While the escape velocity does not represent a physically attainable velocity, it does serve as an upper velocity limit for gas-driven guns.

The relationship between projectile velocity and projectile travel is obtained from the equation of motion,

$$M\frac{du_p}{dt} = P_b A_B . \tag{E-17}$$

Evaluating Eq. E-15 at the projectile base and substituting in Eq. E-17, we have

$$M\frac{du_p}{dt} = P_o A_B\left[1 - \frac{(\gamma - 1)u_p}{2(\gamma R T_o)^{\frac{1}{2}}}\right]^{\frac{2\gamma}{\gamma-1}}$$

or

$$Mu_p \frac{du_p}{dx_p} = P_o A_B \left[1 - \frac{(\gamma - 1) u_p}{2 (\gamma R T_o)^{\frac{1}{2}}} \right]^{\frac{2\gamma}{\gamma - 1}} . \qquad \text{(E-18)}$$

Integrating, we have

$$\int_0^{x_p} \frac{P_o A_B}{M} \, dx_p' = \int_0^{u_p/(\gamma R T_o)^{\frac{1}{2}}} \left[1 - \frac{(\gamma - 1) u_p'}{2 (\gamma R T_o)^{\frac{1}{2}}} \right]^{-\frac{2\gamma}{\gamma - 1}} u_p' \, du_p' \qquad \text{(E-19)}$$

and defining a new variable, $z \equiv u_p / (\gamma R T_o)^{\frac{1}{2}}$, Eq E-19 becomes

$$\frac{P_o A_B x_p}{M \gamma R T_o} = \int_0^z \left[1 - \left(\frac{\gamma - 1}{2} \right) z' \right]^{-\frac{2\gamma}{\gamma - 1}} z' \, dz'$$

which gives us the desired result

$$\frac{P_o A_B x_p}{M \gamma R T_o} = \frac{2}{\gamma + 1} \left\{ \frac{\frac{2}{\gamma - 1} - \frac{\gamma + 1}{\gamma - 1} \left[1 - \frac{\gamma - 1}{2} \frac{u_p}{(\gamma R T_o)^{\frac{1}{2}}} \right]}{\left[1 - \frac{\gamma - 1}{2} \frac{u_p}{(\gamma R T_o)^{\frac{1}{2}}} \right]^{\frac{\gamma + 1}{\gamma - 1}}} + 1 \right\} . \qquad \text{(E-20)}$$

We now introduce reduced variables based on those suggested by Siegel (1979)

$$\overline{x_p} = \frac{P_o A_B x_p}{M\gamma R T_o} \tag{E-21}$$

and

$$\overline{u_p} = u_p(\gamma R T_o)^{-\frac{1}{2}} \tag{E-22}$$

such that Eq. E-20 becomes

$$\overline{x_p} = \frac{2}{\gamma + 1}\left\{\frac{\frac{2}{\gamma - 1} - \frac{\gamma + 1}{\gamma - 1}\left[1 - \frac{\gamma - 1}{2}\overline{u_p}\right]}{\left[1 - \frac{\gamma - 1}{2}\overline{u_p}\right]^{\frac{\gamma + 1}{\gamma - 1}}} + 1\right\}. \tag{E-23}$$

We note now that the reduced projectile travel can be rewritten in terms of the projectile C/M, ε, for finite chamber length x_o,

$$\overline{x_p} = \frac{\varepsilon}{\gamma(1 - \eta\rho_o)} \cdot \frac{x_p}{x_o} \tag{E-24}$$

which makes it possible to relate reduced travel to expansion ratio as defined in Appendix B

$$\varepsilon_f = \frac{x_p + x_o(1 - \eta\rho_o)}{x_o(1 - \eta\rho_o)}. \tag{E-25}$$

Substituting Eq. E-24 in Eq. E-25, we obtain,

$$\varepsilon_f = \frac{x_p}{(1 - \eta\rho_o)x_o} + 1 = \frac{\gamma\overline{x_p}}{\varepsilon} + 1. \tag{E-26}$$

We also note that the reduced projectile velocity (Eq. E-22) can be rewritten in terms of the initial, specific internal energy of the gas (E_o) to give us

$$\overline{u_p} = u_p \left[\gamma(\gamma - 1) E_o\right]^{-\frac{1}{2}} . \qquad \text{(E-27)}$$

These equations are used in the text in plots of ballistic trajectories.

Solution Using Pidduck-Kent Gradient Model

The Pidduck-Kent solution to the Lagrange ballistic problem (i.e., the projectile velocity as a function of gas expansion ratio) is obtained in Appendix B in conjunction with the derivation of the gradient model. We show in this section that this solution can also be obtained in a straightforward manner from the requirement that energy be conserved. The initial, total gas internal energy is

$$E_o = \frac{CRT_o}{\gamma - 1} , \qquad \text{(E-28)}$$

the initial space mean pressure is

$$\overline{P_o} = \left(\frac{1}{\overline{\rho_o}} - \eta\right)^{-1} RT_o = \frac{(\gamma - 1) E_o}{A_B x_o - C\eta} , \qquad \text{(E-29)}$$

and the space mean pressure at any projectile travel (x_p) is

$$\overline{P(x_p)} = \frac{(\gamma - 1) E(x_p)}{A_B (x_o + x_p) - C\eta} \qquad \text{(E-30)}$$

where the internal energy at x_p is obtained by noting that, in the absence of dissipative mechanisms,

$$E(x_p) = E_o - KE_{PROJ} - KE_{GAS} . \qquad \text{(E-31)}$$

As demonstrated in Appendix B, for the Pidduck-Kent gradient model,

$$\overline{P(x_p)} = \left(1 + \frac{\varepsilon}{\delta}\right) P_b(x_p) \tag{E-32}$$

and

$$KE_{GAS} = \left(\frac{\varepsilon}{\delta}\right) KE_{PROJ} . \tag{E-33}$$

Combining these equations, we have

$$E(x_p) = E_o - \frac{1}{2} M u_p^2 \left(1 + \frac{\varepsilon}{\delta}\right) \tag{E-34}$$

or

$$P_b(x_p) = \frac{\gamma - 1}{2}\left[\frac{2E_o}{1 + \frac{\varepsilon}{\delta}} - u_p^2\right]\frac{1}{A_B(x_o + x_p) - C\eta} . \tag{E-35}$$

The equation of motion for the projectile is given by Eq. E-17

$$M\frac{du_p}{dt} = P_b A_B . \tag{E-17}$$

Using Eq. C-33 in Eq. C-17 and rearranging terms, we have

$$u_p \frac{du_p}{dx_p} = \frac{\gamma - 1}{2}\left[\left(\frac{\varepsilon}{1 + \frac{\varepsilon}{\delta}}\right)(2E_o) - u_p^2\right]\frac{1}{x_p + x_o(1 - \eta\rho_o)} \tag{E-36}$$

or

$$\frac{2}{\gamma - 1}\int_0^{u_p} \frac{u_p' \, du_p'}{\left[\left(\frac{\varepsilon}{1 + \frac{\varepsilon}{\delta}}\right)(2E_o) - u_p'^2\right]} = \int_0^{x_p} \frac{dx_p'}{x_p' + x_o(1 - \eta\rho_o)} , \tag{E-37}$$

from which we obtain the desired result

$$u_p^2 = \frac{2\varepsilon E_o}{1 + \frac{\varepsilon}{\delta}} \left[1 - \left(\frac{1}{\varepsilon_f} \right)^{\gamma-1} \right] \tag{E-38}$$

where E_o is the initial, specific internal energy of the gas, and

$$\varepsilon_f \equiv \frac{x_p + x_o(1 - \eta\rho_o)}{x_o(1 - \eta\rho_o)} \tag{E-39}$$

is the expansion ratio of the gas. In the limit of infinite C/M and infinite expansion ratio (Eq. E-38) yields the maximum projectile velocity for the Pidduck-Kent solution

$$\left(u_{max}^{P\text{-}K} \right)^2 = 2(2n + 3) E_o \tag{E-40}$$

or, recalling that $n = \frac{1}{\gamma - 1}$,

$$u_{max}^{P\text{-}K} = \left(\frac{3\gamma - 1}{2\gamma} \right)^{\frac{1}{2}} \left(\frac{2}{\gamma - 1} \right) \left(\gamma R \overline{T_o} \right)^{\frac{1}{2}} . \tag{E-41}$$

Comparing with Eq. E-16, we see that

$$u_{max}^{P\text{-}K} = \left(\frac{3\gamma - 1}{2\gamma} \right)^{\frac{1}{2}} u_{escape} , \tag{E-42}$$

as obtained directly in Appendix B (see Equation B-104) in the derivation of the Pidduck-Kent solution.

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APPENDIX F:

COMPUTER PROGRAM FOR CALCULATION OF PIDDUCK-KENT PARAMETERS

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```
      program pidduck
c
c     Pidduck-Kent constants DEL and A0
c     Note: Accuracy is poor for charge-to-mass ratio < 1E-03
c     Values of DEL and A0 are not calculated for EPS = 0
c     Iteration fails for EPS < 1E-05
c     24 March 1991
c
c     Converted from BASIC to FORTRAN
c     Converter: Caledonia Henry
c     Date: 30 Jan 1992
c
      dimension a(5), d(5)
      data (a(i),i=1,5),(d(i),i=1,5)/-.003659837,-.006938816,
     * .0006852334,-.00003094782,.000000529406,.3667921,-.03838811,
     * .003275623, -.0001455872,.000002498742/
c
      PI=4.0*atan(1.0)
c
c     Open file for output
      open(9,file='pkconst.out')
c
c     Input GAMMA and Charge-to-Mass Ratio
c
  98  write(6,'(a)') "Input Gamma and Charge-to-Mass Ratio"
      read(5,*) gam, eps
c
c     calculate cn and cnn
c
      cn = 1.0/(gam-1.0)
      cnn = cn + 1.0
c
c     Calculate initial estimate for Delta and A0
c     Using Grollman and Baer formula.
c
c     Note: Accuracy decreases rapidly for eps > 20
c
      del = 3.0
      do 100 i = 1,5
      del = del+(a(i)+d(i)/gam)*(eps**i)
 100  continue
c
      if(eps.gt.0.0) go to 200
      az =0.0
      go to 211
c
 200  az1 = (((2.0*cn)+3.0)/del)+((2.0*(cn+1.0))/eps)
      az = 1.0/az1
c
```

```
  211  write(9,'(a)') 'GROLLMAN AND BAER LEAST SQUARES FIT'
       write(9,1000) eps, cn, gam, del, az
 1000  format(1x,"EPS = ",f8.4," N = ",f8.4," GAMMA = ",f7.4," DEL = ",
      *f9.6," A0 = ",f9.6)
c
       if(eps.gt.0.0) go to 224
       azzz = 0.0
       dell = 3.0
       go to 338
c
  224  if(az.lt.0.7) go to 226
       az =0.7
c
c      if A0 > 0.65 then use alternate solution
c
  226  if(az.gt.0.65) go to 250
c
c      Newton-Raphson iteration for A0 <= 0.65
c
  228  j=0
       azz = az
       j = j+1
c
  236  call g0g1g2le( g2, azz, cn, eps,cnn)
       azzz = azz-g2
       if (abs(g2).lt.1.E-06) go to 334
c
       azz = azzz
       if(azz.le.0.65) go to 236
       az = azz
c
c      Newton-Raphson Iteration for A0 > 0.65
c
c      First calculate Beta function - B(0.5,CN+1.0)
c
  250  w = 0.5
       gu = sqrt(PI)
       w = cn + 1.0
       call gamfun(gamma,w)
       gv = gamma
  266  w = cn + 1.5
       call gamfun(gamma,w)
       guv = gamma
       beta = gu*gv/guv
c
c      now proceed with iteration
c
       j=0
       azz=az
```

```
  308  j = j+1
       call g0g1g2gt ( g2, azz, cn, eps, cnn, beta)
       azzz = azz - g2
       if (abs(g2) .lt. 1.E-06) go to 334
       azz = azzz
       if (azz .gt. .65) go to 308
       az = azz
       go to 228
  334  dell = ((1.0/azzz)-((2.0*(cn+1.0))/eps))/(2.0*cn+3.0)
       dell = 1.0/dell
  338  write (9,'(a)') 'CALCULATION**************'
       write (9,1000) eps, cn, gam, dell, azzz
c
       close(9)
       end
c
c
c
       subroutine g0g1g2le ( g2, azz, cn, eps,cnn)
c
c      subroutine for A0 <= 0.65
c
       x=azz
       prod = 1.
       fact = 1.
       sum = 1.
       summ = 0.
       k = 0
  420  k = k + 1
       prod = prod*(k-cn-1.)
       fact = fact * k
       arg = (x**k)/(2.0*k+1.0)
       argg = k*(x**(k-1))/(2.0*k+1.0)
       term = prod*arg/fact
       termm = prod*argg/fact
       sum = sum+term
       summ = summ+termm
       if(abs(term).lt.1.E-10) go to 540
       go to 420
  540  g0=(2.*cnn*(x/((1.-x)**cnn))*sum)-eps
       g1=(g0+eps)*((1./x)+(cnn/(1.-x)))+2.*cnn*(x/((1.0-x)**cnn))*summ
       g2 = g0/g1
       return
       end
c
c
c
       subroutine g0g1g2gt ( g2, azz, cn, eps, cnn, beta)
c
```

```
c       subroutine for A0 > 0.65
c
        x = 1.0-azz
        prod = 1.
        fact = 1.
        sum = 1.
        summ = 0.
        k = 0
 670    k = k + 1
        prod = prod*(k+0.5-1)
        fact = fact*k
        arg = (x**k)*(cnn/(k+cnn))
        argg = k*(x**(k-1))*(cnn/(k+cnn))
        term = prod*arg/fact
        termm = prod*argg/fact
        sum = sum+term
        summ = summ-termm
        if(abs(term).lt.1.E-10) go to 800
        go to 670
 800    ssum = (0.5/azz**0.5)*(beta-((x**cnn)/cnn)*sum)
        ssumm = ((-0.5/azz)*ssum)+((0.5/azz**0.5)*(x**cn)*(sum-(x/cnn)*
       *    *summ))
        g0 = (2.0*cnn*(azz/(x**cnn))*ssum)-eps
        g1 = (g0+eps)*((1.0/azz)+(cnn/x))+2.0*cnn*(azz/(x**cnn))*ssumm
        g2 = g0/g1
        return
        end
c
c
c
        subroutine gamfun(gamma,w)
c
c       gamma function
c
        dimension b(8)
c
        data (b(i),i=1,8)/-.577191652, .988205891, -.897056937,
       *.918206857,-.756704078, .482199394, -.193527818, .035868343/
c
        iw = int(w)
        fw = w-iw
        if(iw.gt.0) go to 920
        gamma = 1.0/fw
        go to 970
 920    gamma =1
        do 960 j = 1,iw-1
        gamma = gamma*(w-j)
 960    continue
 970    gfw = 1
```

```
      do 1000 k = 1,8
      gfw = gfw + (b(k)*(fw**k))
1000  continue
      gamma = gamma*gfw
      return
      end
```

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LIST OF SYMBOLS

A_B	Bore Area
a	Parameter (Eq. B-28)
a_0	Pidduck-Kent Characteristic Parameter (Eq. B-33)
B	Separation Constant (Eq. B-23)
$B_x(a,b)$	Incomplete Beta Function
C	Propellant Charge Mass
$C(x_o)$	Gas Mass Between Breech and x_o (Eq. B-113)
c_v	Specific Heat at Constant Volume
$E_i(t)$	Gas Internal Energy
E_o	Initial Gas Internal Energy
${}_2F_1(a,b;c;z)$	Gauss Hypergeometric Function
$f(x_o)$	Function Describing Spatial Dependence of z (x_o,t) (Eq. B-19)
$f_b \equiv f(y_o)$	Value of $f(x_o)$ at Projectile Base
K, $K(x_o)$	Adiabatic Constant
KE	Kinetic Energy
M	Projectile Mass
n	Polytropic Index, $(\gamma - 1)^{-1}$
P	Gas Pressure
$\overline{P(t)}$	Space Mean Gas Pressure
$P(x_o, t)$	Pressure of Gas Element Initially Located at x_o
$q(\mu)$	Integration Variable (Eq. B-41)
R	Gas Constant
$S(\eta,a_0)$	Integral Arising in Pidduck-Kent Derivation; Specific Gauss Hypergeometric Function (Eq. B-40 and B-41)
t	Time
T	Gas Temperature
$\overline{T(t)}$	Space Mean Gas Temperature
$T(x_o,t)$	Temperature of Gas Element Initially Located at x_o
$u(x_o,t)$	Velocity of Gas Element Initially Located at x_o
u_p	Projectile Velocity
$W(x_o,t)$	Function of Density of Gas Element Initially Located at x_o (Eq. B-12)
$x(x_o,t)$	Position of Gas Element Initially Located at x_o
x_o	Initial Location of Gas Element
$y(t) = x(y_o,t)$	Projectile Position
y_o	Initial Projectile Position
$z(x_o,t)$	Pidduck-Kent Variable, Reduced Gas Element Position (Eq. B-7)
γ	Ratio of Specific Heats

δ	Pidduck-Kent Characteristic Parameter (Eq. B-83)
ε	Ratio of Propellant Charge Mass to Projectile Mass
η	Gas Covolume
$\mu(x_o)$	Integration Variable (Eq. B-66)
ρ	Gas Density
$\rho(x_o,t)$	Density of Gas Element Initially Located at x_o
$\overline{\rho(t)}$	Space Mean Gas Density

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